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The quaternion group and modern physics

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Abstract The paper shows how various physical covariance groups: SO(3), the Lorentz group, the general theory of relativity group, the Clifford algebra, SU(2) and the conformal group can easily be related to the quaternion group. The quaternion calculus is introduced and several physical applications: crystallography, the kinematics of rigid body motion, the Thomas precession, the special theory of relativity, classical electromagnetism, the equation of motion of the general theory of relativity, and Dirac's relativistic wave equation are discussed.

Résumé L'article montre comment plusieurs groupes de covariance de la physique: SO(3), le groupe de Lorentz, le groupe de la relativité générale, l'algèbre de Clifford, SU(2) et le groupe conforme peuvent facilement être reliés au groupe des quaternions. Le calcul quaternionien est introduit et plusieurs applications physiques: la cristallographie, la cinématique du mouvement des corps rigides, la précession Thomas, la théorie de la relativité restreinte, l'équation du mouvement de la relativité générale et l'équation relativiste de Dirac sont traitées.

1. Introduction

Several authors have, in the pages of this journal, advocated the introduction of group theory in physics teaching. Yet, simple presentations together with physical applications seem to be scarce. In the following, I shall show that several of the major covariance groups of physics can easily be related to a finite group, namely, the abstract quaternion group. Most (if not all) of the physical and mathematical results presented in this paper can be found scattered in the literature over a time interval reaching into the 19th century. After introducing the quaternion group and the quaternion algebra, the paper analyses the connection of various major covariance groups of physics with the quaternion group together with important physical applications.

2. The quaternion group and the quaternion algebra

The abstract quaternion group (denoted Q) which was discovered by William Rowan Hamilton (Hankins 1980, Crowe 1967a, Hamilton 1931–67, Graves 1882–9) in 1843, is constituted by the eight elements ±1, ±i, ±j, ±k, satisfying the relations

$$\mathrm{i}^2=\mathrm{j}^2=\mathrm{k}^2=\mathrm{ijk}=-1$$

or, more explicitly

$$\mathrm{i}^2=\mathrm{j}^2=\mathrm{k}^2=-1$$
$$\mathrm{ij}=-\mathrm{ji}=\mathrm{k}$$
$$\mathrm{jk}=-\mathrm{kj}=\mathrm{i}$$
$$\mathrm{ki}=-\mathrm{ik}=\mathrm{j}$$

1 is the unit element. The subgroups of Q are:

(1)
(1, −1)
(1, −1, i, −i)
(1, −1, j, −j)
(1, −1, k, −k).

To obtain the quaternion algebra (denoted $\boldsymbol{H}$), consider the vector space of ordered sets of four numbers (real or complex): a, b, ... called quaternions‡

$$a=a_0 1+a_1\mathrm{i}+a_2\mathrm{j}+a_3\mathrm{k}$$
$$=(a_0, a_1, a_2, a_3)$$
$$=(a_0, \boldsymbol{a}).$$

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‡ For a complete bibliography on works on quaternions up to 1912, see Macfarlane (1904) and the *Bulletin* (1900–13)

This vector space is transformed into an associative quaternion algebra via the multiplication rule

$$\begin{aligned} ab = &(a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) \\ &+ (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)\mathrm{i} \\ &+ (a_0b_2 + a_2b_0 + a_3b_1 - a_1b_3)\mathrm{j} \\ &+ (a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1)\mathrm{k} \end{aligned}$$

which is obtained from the multiplication table of Q. The above multiplication rule can easily be remembered by expressing it in terms of the ordinary dot and vector products

$$ab = (a_0b_0 - \boldsymbol{a} \cdot \boldsymbol{b},\ a_0\boldsymbol{b} + b_0\boldsymbol{a} + \boldsymbol{a} \times \boldsymbol{b})$$

if $\boldsymbol{a} \times \boldsymbol{b} \neq 0$, the quaternion product is noncommutative. The presence of the dot and vector products in the quaternion product is no coincidence since, historically, W J Gibbs (Crowe 1967b, Bork 1966) introduced these two products by splitting the quaternion product and by taking $a_0 = b_0 = 0$. It is to be noticed that the quaternion algebra is distinct from the algebra of the quaternion group which is eight-dimensional. The real quaternion algebra $\boldsymbol{H}(\boldsymbol{R})$, which contains $\boldsymbol{R}$ and $\boldsymbol{C}$ as particular cases, has a unique place among all associative algebras over $\boldsymbol{R}$ since it is the only finite-dimensional division algebra (Dickson 1914) over $\boldsymbol{R}$, besides $\boldsymbol{R}$ and $\boldsymbol{C}$. I shall use both $\boldsymbol{H}(\boldsymbol{R})$ and $\boldsymbol{H}(\boldsymbol{C})$. The latter is still an associative algebra but not a division algebra.

Considering a quaternion $q = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}$, I shall use the following terminology (Hamilton 1969a):

$$Sq = q_0$$

is the scalar part of q;

$$Vq = q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}$$

is the vector part of q;

$$q_c = q_0 - q_1\mathrm{i} - q_2\mathrm{j} - q_3\mathrm{k}$$

is the quaternion conjugate† of q;

$$q^* = q_0^* + q_1^*\mathrm{i} + q_2^*\mathrm{j} + q_3^*\mathrm{k}$$

is the complex conjugate of q;

$$\begin{aligned} Nq &= qq_c = q_cq \\ &= q_0^2 + q_1^2 + q_2^2 + q_3^2 \end{aligned}$$

is the norm of q. Among frequently used relations are:

$$\begin{aligned} &(q_c^*) \equiv (q_c)^* = (q^*)_c \\ &q^{-1} = q_c/qq_c \\ &(q_1q_2)_c = q_{2c}q_{1c} \\ &S(q_1q_2) = S(q_2q_1) \\ &S(q_1q_2q_3) = S(q_2q_3q_1) = S(q_3q_1q_2) \end{aligned}$$

and the important relation

$$N(q_1q_2) = (Nq_1)(Nq_2).$$

The differential (Hamilton 1969b) of a quaternion product is given by the relation

$$\mathrm{d}(q_1q_2) = (\mathrm{d}q_1)q_2 + q_1(\mathrm{d}q_2)$$

where the order of the terms has to be respected. Finally, it is to be kept in mind that a scalar commutes with any quaternion.

† Hamilton wrote Kq instead of q_c.

3. Representations of a group

By definition, a representation of a group G is a mapping of G onto a group T of operators (acting on a vector space) which preserve the group law, i.e., are such that

$$T(g_1)T(g_2) = T(g_1g_2)$$

for any g_i of G. The representation is said to be linear if the operators are linear.

The theory of representations of groups allows the precise definition of the concept of a physical quantity (Kustaanheimo 1955). To this effect, consider an abstract group G and a representation T of it acting on a vector space V. If T is a transitive transformation group (Iyanaga and Kawada 1977) of V, then the elements of V are called physical quantities with respect to G. The group T is called the covariance group. As illustrations, consider the following examples: if G is the abstract three-dimensional rotation group and T a representation of it, the elements of V are called vectors (in the physical sense); if G is the general linear group, the elements of V are called tensors; if G is the Clifford group, the elements of V are called spinors. Once a physical quantity is defined, physical equations are obtained by combining physical quantities with respect to the same group and representation.

As an example of a quaternion representation (Du Val 1964b), consider the transformations T given by

$$q' = aqb$$

where a, b, q, q' are complex quaternions (of norm $\neq 0$). These transformations constitute a group, the product of two transformations T_2T_1 being defined by

$$q'' = a_2q'b_2 = a_2a_1qb_1b_2.$$

The inverse transformation is given by $q = a^{-1}q'b^{-1}$. Take for G the group $\boldsymbol{H} \times \boldsymbol{H}$ of ordered sets of two quaternions (of norm $\neq 0$) with the multiplication rule

$$(a_2, b_2) \times (a_1, b_1) = (a_2a_1, b_1b_2).$$

Since

$$T_2(a_2, b_2)T_1(a_1, b_1) = T(a_2a_1, b_1b_2)$$

it follows that T is a representation of $\boldsymbol{H} \times \boldsymbol{H}$. This representation yields several of the major covariance groups of physics as particular cases, as we shall now see, together with various physical applications of the quaternion formalism.

4. The rotation group SO(3)

The three-dimensional rotation group SO(3) is the group of transformations which leave the quantity $x^2+y^2+z^2$ invariant. Any such transformation can be expressed in the quaternion form (Brand 1947)

$$q' = rqr_c$$

where r is a quaternion such that $rr_c = 1$, and where $q = x\mathrm{i}+y\mathrm{j}+z\mathrm{k}$, $q' = x'\mathrm{i}+y'\mathrm{j}+z'\mathrm{k}$. Reciprocally, any transformation $q' = rqr_c$ (with $rr_c = 1$) is a rotation. The invariance of the norm of q results from the relation

$$q'q'_c = (rqr_c)(rqr_c)_c = qq_c = x^2+y^2+z^2.$$

The inverse transformation is given by $q = r_c q' r$; the composition of two rotations is given by the rule

$$q' = r_2(r_1 q r_{1c})r_{2c} = (r_2 r_1)q(r_2 r_1)_c.$$

Hence, SO(3) constitutes a representataion of $\boldsymbol{H}(\boldsymbol{R})\times\boldsymbol{H}(\boldsymbol{R})$. Since, generally, $r_2 r_1 \neq r_1 r_2$, it follows that the composition of three-rotations is generally noncommutative. If r is written in the form

$$r = (\cos\theta, \sin\theta\boldsymbol{e})$$

where $\boldsymbol{e}$ is a unit vector (i.e. $N\boldsymbol{e} = 1$), then the transformation $q' = rqr_c$ corresponds to a conical rotation of Vq by 2θ around the vector $\boldsymbol{e}$. The scalar part of q remains invariant under a rotation since

$$Sq' = Sr_c rq = Sq.$$

As physical applications of SO(3), we shall consider crystallography and the kinematics of rigid body motion.

4.1. Crystallography

If r takes successively all the values of a finite subgroup of real quaternions, the transformations $q' = rqr_c$ will constitute a subgroup of SO(3) (r and $-r$ leading to the same rotation). The finite subgroups of real quaternions (Stringham 1881, Shaw 1907a) are of five types:

(i) cyclic groups (order of the group: $N = r$)

$$m^{4n/r} = (\cos 2\pi/r, \sin 2\pi/r\boldsymbol{e})^n = (\cos 2\pi n/r, \sin 2\pi n/r\boldsymbol{e})$$

with $n = 1 \ldots r$ and where $\boldsymbol{e}$ is any unit vector;

(ii) double-dihedral groups ($N = 4r$)

$$m^{4n/r}p^h$$

with $S(pm) = 0$, $p^2 = -1$, and $n = 1 \ldots r$, $h = 1 \ldots 4$;

(iii) double-tetrahedral group ($N = 24$)

$$\pm1, \pm\mathrm{i}, \pm\mathrm{j}, \pm\mathrm{k}$$
$$\tfrac{1}{2}(\pm1\pm\mathrm{i}\pm\mathrm{j}\pm\mathrm{k});$$

(iv) double-octahedral group ($N = 48$)

$$\pm1, \pm\mathrm{i}, \pm\mathrm{j}, \pm\mathrm{k}$$
$$\tfrac{1}{2}(\pm1\pm\mathrm{i}\pm\mathrm{j}\pm\mathrm{k}), 2^{-1/2}(\pm1\pm\mathrm{i})$$
$$2^{-1/2}(\pm1\pm\mathrm{j}), 2^{-1/2}(\pm1\pm\mathrm{k})$$
$$2^{-1/2}(\pm\mathrm{i}\pm\mathrm{j}), 2^{-1/2}(\pm\mathrm{j}\pm\mathrm{k})$$
$$2^{-1/2}(\pm\mathrm{k}\pm\mathrm{i});$$

(v) double-icosahedral group† ($N = 120$)

$$\pm\mathrm{k}^{2n/5}, \pm\mathrm{j}\mathrm{k}^{2n/5}$$
$$\pm\frac{\mathrm{k}^{2n/5}(\mathrm{i}+\omega\mathrm{k})\mathrm{k}^{2s/5}}{(1+\omega^2)^{1/2}}$$
$$\pm\frac{\mathrm{k}^{2n/5}(\mathrm{i}+\omega\mathrm{k})\mathrm{j}\mathrm{k}^{2s/5}}{(1+\omega^2)^{1/2}}$$

with n, $s = 1 \ldots 5$, $\omega = 2\cos 72° = \tfrac{1}{2}(-1+5^{1/2})$. These groups, together with the rotation transformation $q' = rqr_c$ and the inversion transformation $q' = q_c$, yield all 32 classes of crystals (Shaw 1907b, 1922).

4.2. Kinematics of rigid body motion

The most general motion of a rigid body being a combination of a translation and a rotation, it can therefore be represented by the transformation (Tait 1890a, Laisant 1877)

$$q' = s + rqr_c \tag{1}$$

with $rr_c = 1$ and where s corresponds to the translation (r and s are functions of the time t). By taking the derivative with respect to the time, one obtains

$$\mathrm{d}q'/\mathrm{d}t = [\mathrm{d}s/\mathrm{d}t + (\mathrm{d}r/\mathrm{d}t)qr_c + rq(\mathrm{d}r_c/\mathrm{d}t)] + r(\mathrm{d}q/\mathrm{d}t)r_c$$

or

$$\mathrm{d}q'/\mathrm{d}t = [\mathrm{d}s/\mathrm{d}t + \tfrac{1}{2}(\omega rqr_c - rqr_c\omega)] + r(\mathrm{d}q/\mathrm{d}t)r_c = [\mathrm{d}s/\mathrm{d}t + V(\omega rqr_c)] + r(\mathrm{d}q/\mathrm{d}t)r_c$$

where $\omega = 2(\mathrm{d}r/\mathrm{d}t)r_c$ is the instantaneous angular velocity quaternion (ω is a vector since $S\omega = 0$, because of the relation $rr_c = 1$). If q, for example, corresponds to the coordinates with respect to a system rigidly attached to the body and q' to the coordinates with respect to a fixed inertial system, then the quantity in square brackets corresponds to the velocity of a fixed point of the rigid body and the second term to the relative velocity. A second differentiation of equation (1) yields

† Compare with Klein (1913).

$$d^2q'/dt^2 = [d^2s/dt^2 + (d^2r/dt^2)qr_c + rq(d^2r_c/dt^2) + 2(dr/dt)q(dr_c/dt)] + 2[(dr/dt)(dq/dt)r_c + r(dq/dt)(dr_c/dt)] + r(d^2q/dt^2)r_c.$$

The quantity in the first square brackets corresponds to the acceleration of a fixed point of the rigid body; the second term corresponds to the Coriolis acceleration, and the last term to the relative acceleration. It seems difficult to conceive a simpler derivation of this useful relation.

5. The Lorentz group

The Lorentz group is the group of homogeneous transformations which conserve the quantity $c^2t^2 - x^2 - y^2 - z^2$. If there is no time reversal and no space reflection, the transformations are said to be orthochronous proper. Any orthochronous proper Lorentz transformation can be expressed by a quaternion transformation (Synge 1972b, Cailler 1917) of the type

$$q' = aqa_c^* \tag{2}$$

where a is a complex quaternion such that $aa_c = 1$, and where $q = (ct, ix, iy, iz)$, $q' = (ct', ix', iy', iz')$†. I shall refer to quaternions such that $q_c^* = q$ as minquats (minkowskian quaternions)‡. The above transformation has the usual six parameters and conserves the norm since

$$q'q'_c = (aqa_c^*)(aqa_c^*)_c = qq_c.$$

It also preserves the minquat character of q since

$$q'^*_c = (aqa_c^*)_c^* = aqa_c^* = q'.$$

The composition of two Lorentz transformations follows the rule

$$q' = a_2[a_1q(a_1)_c^*](a_2)_c^* = (a_2a_1)q(a_2a_1)_c^* = a_3q(a_3)_c^*$$

with $a_3 = a_2a_1$. Since the transformations (2) do form a group, and from the considerations above, it follows that the Lorentz group constitutes a representation of $\boldsymbol{H}(\boldsymbol{C}) \times \boldsymbol{H}(\boldsymbol{C})$.

A pure Lorentz transformation is a Lorentz transformation which contains no rotation and is of the type

$$q' = bqb_c^*$$

where b is a minquat such that $bb_c = 1$. A general Lorentz transformation is obtained by combining a pure Lorentz transformation bqb_c^* with a three-dimensional rotation† rqr_c. This can be done in two ways since $q' = b(rqr_c)b_c^*$ is generally different from $q' = r(bqb_c^*)r_c$.

Reciprocally, any general Lorentz transformation $q' = aqa_c^*$ can be decomposed into a pure Lorentz transformation and a rotation. The problem simply consists in finding two quaternions b and r such that $a = br$ ($a = r'b'$) where b (b') is a unit minquat (i.e. a minquat of norm equal to 1) and r (r') a real unit quaternion ($rr_c = 1$, $r'r'_c = 1$). The equation $a = br$ (and similarly $a = r'b'$) with the above conditions is solved as follows (Conway 1948, 1953b). Since

$$a_c^* = r_c^*b_c^* = r_cb$$

one has $aa_c^* = b^2$, which is an equation of the type $d = b^2$, where d is a unit minquat. To solve this equation, write

$$2b^2 = 2d$$
$$b^2d = d^2$$
$$b^2d_c = 1$$

add, and obtain

$$b^2(2 + d + d_c) = (1 + d)^2.$$

The solution is thus given by

$$b = \pm(1 + d)/|N(1 + d)|^{1/2}$$

where $d = aa_c^*$ in the case here considered, and where the vertical bars stand for the absolute value. Finally, one verifies that this is indeed a solution. The quaternion r of the rotation is given by

$$r = b_ca = \pm(a + a^*)/|N(1 + aa_c^*)|^{1/2}.$$

The problem of the reduction of a Lorentz transformation into a pure Lorentz transformation and a rotation is thus solved in the most general case. As an immediate application, consider a Lorentz transformation obtained from the composition of two pure Lorentz transformations. Since $a = b_1b_2$ will in general be a complex quaternion (not a minquat), one will generally have $a = br$. Hence, the resulting Lorentz transformation will generally contain a rotation; this is the essence of the Thomas precession effect (Thomas 1926, 1927). As further applications of the Lorentz group, we shall now briefly discuss the special theory of relativity and classical electromagnetism.

5.1. *The special theory of relativity*

Consider the pure Lorentz transformation

$$q' = bqb_c^*$$

† From now on, and in the rest of this paper, bold face type will be used for the elements $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ of the abstract quaternion group; i in ordinary roman type represents the complex imaginary $(-1)^{1/2}$.

‡ The term is used by Synge (1972a). In contradistinction to Synge, I shall adopt the form $q = (ct, ix, iy, iz)$ instead of $q = (ict, x, y, z)$, in order to have $ds^2 = dq\, dq_c > 0$.

† The previous results concerning rotations are not affected by the introduction of minquats.

with

$$b = (\cosh\phi, \mathrm{i}\sinh\phi\boldsymbol{e})$$
$$q = (ct, \mathrm{i}x, \mathrm{i}y, \mathrm{i}z)$$
$$q' = (ct, \mathrm{i}x', \mathrm{i}y', \mathrm{i}z')$$

and where $\boldsymbol{e}$ is a real unit vector. The four-veclocity transforms according to the relation

$$u' = bub_c^*$$

where $u = \mathrm{d}q/\mathrm{d}s$, $u' = \mathrm{d}q'/\mathrm{d}s$, and $\mathrm{d}s^2 = \mathrm{d}q\,\mathrm{d}q_c$. If $u = 1$, one has

$$u' = bb_c^* = (\cosh\phi, \mathrm{i}\sinh\phi\boldsymbol{e})^2 = (\cosh 2\phi, \mathrm{i}\sinh 2\phi\boldsymbol{e}).$$

If, furthermore,

$$\cosh 2\phi = (1 - v^2/c^2)^{-1/2} = \gamma$$
$$\tanh 2\phi = -v/c$$

then

$$u' = (\gamma, -\mathrm{i}\gamma v/c\boldsymbol{e}).$$

Thus, the above transformation transforms a point at rest into a point moving with a speed v in the direction $-\boldsymbol{e}$; it corresponds to a standard Lorentz transformation (from a passive point of view) in the $\boldsymbol{e}$ direction.

The four-momentum of a particle of rest mass m is defined by $p = mu$ and satisfies the relation $pp_c = m^2$, since $uu_c = 1$. The four-angular momentum of a material point can be defined by

$$L = V(qp_c) = [0, \boldsymbol{r}\times\boldsymbol{p} + \mathrm{i}(E\boldsymbol{r}/c - ct\boldsymbol{p})].$$

Under an arbitrary Lorentz transformation, L transforms according to the rule

$$\begin{aligned} L' &= V(q'p_c') = V(aqa_c^*a^*p_ca_c) \\ &= V\{a[S(qp_c) + V(qp_c)]a_c\} \\ &= V(aLa_c) \\ &= aLa_c \end{aligned}$$

where the last equation follows from the relation $S(aLa_c) = 0$. Hence, the transformation rule of L yields the relativistic invariant

$$LL_c = (\boldsymbol{r}\times\boldsymbol{p})^2 - (E\boldsymbol{r}/c - ct\boldsymbol{p})^2.$$

5.2. Classical electromagnetism

Consider the four vector potential $A = (\phi/c, \mathrm{i}\boldsymbol{A})$, the relativistic operator

$$D = \left(\frac{\partial}{c\,\partial t}, -\mathrm{i}\boldsymbol{\nabla}\right)$$

and the quaternion conjugate operator

$$D_c = \left(\frac{\partial}{c\,\partial t}, \mathrm{i}\boldsymbol{\nabla}\right).$$

Under a Lorentz transformation, D transforms† like

$$q = (ct, \mathrm{i}x, \mathrm{i}y, \mathrm{i}z)$$
$$D' = aD(\ )a_c^*.$$

The quaternion

$$F = D_cA = \left[\frac{\partial\phi}{c^2\,\partial t} + \operatorname{div}\boldsymbol{A}, -\operatorname{rot}\boldsymbol{A} + \mathrm{i}\left(\frac{\partial\boldsymbol{A}}{c\,\partial t} + \operatorname{grad}(\phi/c)\right)\right]$$

is equal to

$$F = (0, -\boldsymbol{B} - \mathrm{i}\boldsymbol{E}/c)$$

if one adopts the Lorentz gauge $S(D_cA) = 0$ and the usual definitions of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively. Under a Lorentz transformation, F transforms according to $F' = a^*Fa_c^*$, which yields the relativistic invariant

$$F'F_c' = FF_c = (\boldsymbol{B}^2 - \boldsymbol{E}^2/c^2) + 2\mathrm{i}\boldsymbol{B}\cdot\boldsymbol{E}/c.$$

Furthermore, since

$$DD_c = D_cD = -\Box = \frac{\partial^2}{c^2\,\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

Maxwell's eight equations for the vacuum are given by one quaternion equation (Silberstein 1914, Cailler 1917, Hermann 1974, Imaeda 1976, Edmonds 1978), namely

$$DD_cA = \mu_0 J \qquad (3)$$

or

$$DF = \mu_0 J$$

where $J = (\rho c, \mathrm{i}\rho\boldsymbol{v})$ is the four current density. Maxwell's equations can also be written in the form

$$D_c(DA_c) = \mu_0 J_c$$

where $DA_c = (0, -\boldsymbol{B} + \mathrm{i}\boldsymbol{E}/c)$.

If one applies the operator D_c to equation (3) one obtains

$$D_c(DD_cA) = \mu_0 D_cJ.$$

Since $S(D_cJ) = 0$ expresses the conservation of electric charge, it follows that the gauge condition has to satisfy the equation

$$SD_c(\Box A) = S\Box(D_cA) = \Box(SD_cA) = 0$$

which is less restrictive than the Lorentz gauge $S(D_cA) = 0$. If $D_cJ = 0$, one has the relation

$$D_c(DF) = [0, \Box(\boldsymbol{B} + \mathrm{i}\boldsymbol{E}/c)] = 0.$$

† To prove this, use the relation

$$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial}{\partial x^{\alpha}} \cdot \frac{\partial x^{\alpha}}{\partial x'^{\mu}}$$

write out $q' = aqa_c^*$, $q = a_cq'a^*$, and compare the coefficients.

The equation of motion of a particle with an electric charge ε in an electromagnetic field is given by

$$dp/ds = \tfrac{1}{2}\varepsilon[(DA_c)u - u(D_cA)]$$

where $p = mu$ is the four-momentum of the particle. The relativistic covariance of the equation of motion is manifest since a quaternion of the type fg_ch transforms like q.

6. The general theory of relativity (GTR) group

The GTR group† is simply the generalised Lorentz group

$$u' = aua_c^* \qquad (4)$$

where a is an arbitrary function such that $aa_c = 1$ and where u is a minquat. At any given space–time point, the GTR group reduces to the Lorentz group. By differentiating relation (4), one obtains

$$du'/ds = (da/ds)ua_c^* + au(da_c^*/ds) + a(du/ds)a_c^* \qquad (5)$$

where $ds^2 = dq\, dq_c$ and $dq = (c\, dt, i\, dx, i\, dy, i\, dz)$.

As an application, consider u' to be the four-velocity of a particle in its instantaneous rest frame (thus $du'/ds = 0$) and u to be the four-velocity of the particle with respect to a fixed inertial system. After rearrangement (and taking into account the relation $a_c\, da = -da_c a$, since $aa_c = 1$), relation (5) yields

$$du/ds = -[(a_c\, da/ds)u - u(a_c\, da/ds)^*].$$

This equation corresponds to the equation of motion of the general theory of relativity. If the function $a_c\, da/ds$ is known, the motion is determined. Newton's law of gravitation, for example, is obtained by considering the nonrelativistic limit and by taking

$$2a_c\, da/ds = [0, (cr^2)^{-1}(\boldsymbol{L} + i\boldsymbol{A}/c)]$$

where $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{v}$ is the angular momentum per unit mass, and where

$$\boldsymbol{A} = (\boldsymbol{v} \times \boldsymbol{L} - GM\boldsymbol{r}/r)$$

is the misnamed Runge–Lenz vector (Hamilton 1969c, Tait 1890b, Goldstein 1975, 1976) per unit mass of the particle.

7. The Clifford algebra

The Clifford algebra (Shaw 1907c, Clifford 1878) is the algebra with n generators $e_1 \ldots e_n$, such that

$$e_i^2 = -1, \qquad e_ie_j + e_je_i = 0$$

with $i, j = 1 \ldots n$ $(i \neq j)$, and

$$e_i(e_je_k) = (e_ie_j)e_k$$

with $i, j, k = 1 \ldots n$. The order of the algebra is 2^n.

† For a historical analysis, see Girard 1981.

The Clifford algebra is directly linked to quaternions via the following theorem (Shaw 1907c, Clifford 1968b).

Theorem: If $n = 2m$ (m integer), the Clifford algebra (of order $2^n = 4^m$) is the direct product of m quaternion algebras. If $n = 2m + 1$, the Clifford algebra (of order $2^n = 2^{2m+1}$) is the direct product of m quaternion algebras and the algebra

$$\omega_0^2 = \omega_0 = \omega_1$$
$$\omega_0\omega_1 = \omega_1\omega_0 = \omega_1.$$

As illustrations of the theorem, we shall consider the cases for $n = 2, 3, 4$.

$n = 2$: The Clifford algebra contains the four elements -1, e_1, e_2, e_1e_2, and is isomorphic to the quaternion algebra 1, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ over $\boldsymbol{R}$ or $\boldsymbol{C}$, the elements e_1, e_2 being respectively associated with $\boldsymbol{i}$ and $\boldsymbol{j}$.

$n = 3$: The Clifford algebra is isomorphic to the algebra given by the direct product

$$(1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}) \times (\omega_0, \omega_1)$$

the generators e_1, e_2, e_3 of the Clifford algebra being respectively associated with $\boldsymbol{i}\omega_0$, $\boldsymbol{j}\omega_0$, $\boldsymbol{k}\omega_1$. The general element of this algebra can be written in the form $q + \omega p$, where q and p are quaternions and where ω satisfies the relation $\omega^2 = 1$ and commutes with any quaternion. Clifford was to call the element $q + \omega p$ a biquaternion (Clifford 1968a, 1968c)‡.

$n = 4$: This Clifford algebra is isomorphic to the algebra obtained by forming the direct product

$$(E_1, \boldsymbol{i}_1, \boldsymbol{j}_1, \boldsymbol{k}_1) \times (E_2, \boldsymbol{i}_2, \boldsymbol{j}_2, \boldsymbol{k}_2)$$

where E stands for the unit element of the quaternion group; the four generators e_1, e_2, e_3, e_4 of the Clifford algebra correspond respectively to $\boldsymbol{i}_1E_2$, $\boldsymbol{j}_1E_2$, $\boldsymbol{i}_2E_1$, $\boldsymbol{j}_2E_1$. As is well known, Dirac's matrices constitute a representation of this Clifford algebra. Expressed in their standard form (Flügge 1974a), the Dirac matrices γ_1, γ_2, γ_3, γ_4, $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ are respectively equivalent to the quaternion operators (Conway 1953a, Synge 1972c):

$$i(\ )\boldsymbol{i},\ \boldsymbol{j}(\ )\boldsymbol{k},\ i(\ )\boldsymbol{j},\ \boldsymbol{k}(\ )\boldsymbol{k},\ \boldsymbol{i}(\ )\boldsymbol{k}$$

operating on the complex function

$$\psi = \psi_0 + \psi_1\boldsymbol{i} + \psi_2\boldsymbol{j} + \psi_3\boldsymbol{k};$$

the composition of two operators, say $\boldsymbol{j}(\ )\boldsymbol{k}$ followed by $i(\ )\boldsymbol{i}$, is given by the usual rule: $i\boldsymbol{j}(\ )\boldsymbol{ki}$. Dirac's relativistic wave equation (Flügge 1974b) can thus be written in the quaternion form

$$(iD_1)\psi\boldsymbol{i} + (\boldsymbol{j}D_2)\psi\boldsymbol{k} + (iD_3)\psi\boldsymbol{j} + (\boldsymbol{k}D_0)\psi\boldsymbol{k} + \chi\psi = 0$$

where:

‡ Clifford's biquaternions are distinct from Hamilton's biquaternions; the latter are simply complex quaternions.

$$D\mu = \frac{\partial}{\partial x^{\mu}} - (ie/\hbar c)A\mu$$
$$A_0 = i\phi$$
$$e\phi = V$$
$$D_0 = -(i/c)(\partial/\partial t).$$

More symmetrical forms are possible (Conway 1953c, Edmonds 1974). The function ψ is an example of a spinor in four dimensions. More generally, the Clifford algebra allows the definition of spinors in n dimensions (Chevalley 1954, Waerden 1974).

8. Other groups

As further and final applications of quaternions to physics, let us briefly mention SU(2) and the conformal group.

8.1. The unitary group SU(2)

The unitary group SU(2) is the group of 2×2 unitary matrices† of determinant 1. This group is isomorphic to the group of real quaternions via the correspondence (Du Val 1964a)

$$\mathbf{A} = \begin{pmatrix} a_0 + ia_3 & -ia_1 + a_2 \\ -ia_1 - a_2 & a_0 - ia_3 \end{pmatrix} \leftrightarrow q = (a_0, a_1, a_2, a_3)$$

with $\det \mathbf{A} = Nq = 1$.

8.2. The conformal group

It is well-known that Maxwell's equations in vacuum are covariant not only with respect to the Lorentz group but also with respect to the conformal group. This 15 (real) parameter group can be defined (Bessel-Hagen 1921) as the group of transformations such that if $N(\mathrm{d}q) = 0$, with $\mathrm{d}q = (c\,\mathrm{d}t, \mathrm{i}\,\mathrm{d}x, \mathrm{i}\,\mathrm{d}y, \mathrm{i}\,\mathrm{d}z)$, then $N(\mathrm{d}q') = 0$. Consequently, the group contains in particular the space–time translations, the Lorentz transformations, and the dilatations $q' = \lambda q$, where λ is a constant. The transformations relative to the four remaining parameters can be expressed in quaternion form by the formulae‡:

$$q' = (1 + qa_c)^{-1}q = q(1 + a_c q)^{-1}$$
$$= (q + aqq_c)/[1 + 2S(qa_c) + aa_c qq_c] \qquad (6)$$

where q, q' are minquats and where a_c is a constant minquat. The above formulae can also be written

$$(q')^{-1} = q^{-1} + a_c. \qquad (7)$$

By using the quaternion formula $(ab)^{-1} = b^{-1}a^{-1}$ it is easy to show that the conformal transformations given by equation (6) form a group, the inverse transformation being given by

$$q = (1 - q'a_c)^{-1}q'.$$

Furthermore, one has the relations §:

$$qa_c q' = q'a_c q$$
$$N(\mathrm{d}q') = (Nq'/Nq)^2 N(\mathrm{d}q)$$
$$Nq' = Nq/N(1 + qa_c)$$
$$\mathrm{d}q' = (1 + qa_c)^{-1}\mathrm{d}q(1 + a_c q)^{-1}.$$

From the second relation, it follows that the transformations here considered are indeed conformal transformations.

9. Conclusion

The paper has related major covariance groups of physics to the quaternion group and has given several physical applications, most of them at undergraduate level. It is hoped that the treatment of the representations of the quaternion group will have shown that quaternion group theory is quite accessible to undergraduate students and might further their comprehension of physics by exhibiting the deep unity of physical phenomena.

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† A unitary matrix is a complex matrix $\mathbf{A}$ such that $\mathbf{A}^{-1} = (\mathbf{A}^{t})^{*}$, where $(\mathbf{A}^{t})^{*}$ is the complex conjugate matrix transpose of $\mathbf{A}$.

‡ Compare with Gürsey *et al* 1980.

§ To derive the second equation, differentiate equation (7) and use the relation $\mathrm{d}(q^{-1}) = -q^{-1}(\mathrm{d}q)q^{-1}$ which follows from the differentiation of the relation $q^{-1}q = qq^{-1} = 1$.

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