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DYNAMICS AND CONDENSATION OF COMPLEX SINGULARITIES FOR BURGERS' EQUATION II*

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Abstract. The zero-viscosity limit of a meromorphic solution to Burgers' equation (BE) is found via an integral representation of the Mittag–Leffler expansion of the solution involving a "polar" measure. The weak zero-viscosity limit of this Borel measure (analogously to the zero-dispersion limit of the spectral measure in the Korteweg–de Vries (KdV) problem) corresponds to the asymptotic density of poles which characterizes their condensation on the imaginary axis. The resulting integral representation of the inviscid solution is computed by residues and is shown to match the characteristic solution up to the inviscid shock time t_*. The continuum limit of the Mittag–Leffler expansion and the Calogero dynamical system (CDS) (which describes the time evolution of the poles) is a system of two integro-differential equations which provide a new representation of the solution to the inviscid BE. For $t \leq t_*$, a uniform asymptotic expansion of the Fourier transform of the inviscid solution is obtained, thereby providing the analyticity properties of the inviscid solution.

Key words. partial differential equations, zero-viscosity limit, pole condensation

AMS subject classifications. 35A20, 35A40, 35B40, 35Q53, 41A60

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1. Introduction.

In this article we continue the investigation from part I [18] of the spatial analyticity properties of a solution to Burgers' equation (hereafter referred to as "BE"):

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = \nu\,\frac{\partial^2 u}{\partial x^2}, \qquad x \in \mathbb{R},\ t > 0,\ \nu \geq 0. \tag{1.1}$$

Previous work concerning the analyticity properties of BE can be found in [3, 4, 13, 14, 18, 20].

We focus on a particular initial value problem (IVP) for (1.1) which was introduced by Fournier and Frisch [13] and further studied by Bessis and Fournier in [3, 4]. In this problem, the initial condition is given by

$$u(x, 0) = u_0(x) = 4x^3 - x/t_*, \qquad x \in \mathbb{R}, \tag{1.2}$$

where t_* is a fixed positive parameter. This initial value is chosen for its generic property, which is due to the type of singularity occurring in the inviscid solution ($\nu = 0$) at the shock time $t = -\left(\inf_x u_0'(x)\right)^{-1} = t_*$ (cf. Appendix A for more details).

It was shown in part I that all meromorphic solutions to BE with an odd initial data must have a symmetric pole expansion of the form

$$u_\nu(x, t) = \frac{x}{t} - 2\nu \sum_{n\in\mathcal{I}} \frac{1}{x - a_n(t, \nu)}, \tag{1.3}$$

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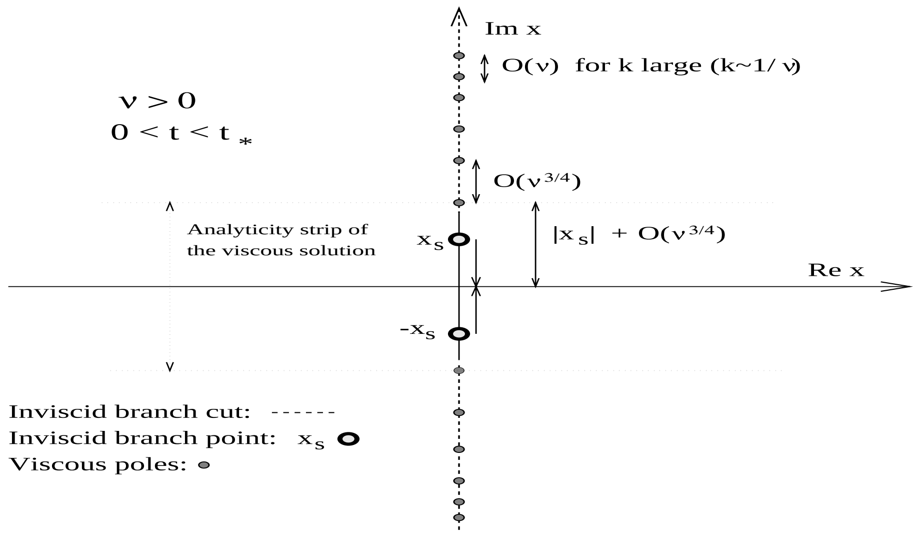

FIG. 1.1. *Inviscid branch points, branch cuts, and viscous poles for $\nu > 0$ and $0 < t < t_*$. The poles are located above the inviscid branch-point singularities according to the asymptotic formula $\beta_k(t,\nu) = \Im x_s(t) + \mathcal{O}(\nu^{3/4})$ as $\nu \to 0^+$ for k fixed. The distance separating two successive poles is asymptotically given by $\Delta\beta_k = \mathcal{O}(\nu)$ as $\nu \to 0^+$ for k large ($k \sim 1/\nu$).*

where $\mathcal{I} \subseteq \mathbb{Z}$ is a finite or countable symmetric set (i.e., if $a_n \in \mathcal{I}, a_{-n} = -a_n \in \mathcal{I}$). Moreover, the poles $\{a_n(t,\nu)\}_{n\in\mathcal{I}}$ must satisfy a Calogero-type dynamical system [8] (hereafter referred to as "CDS") of the form

$$\frac{da_n}{dt} = \frac{a_n}{t} - 2\nu \sum_{\substack{l\in\mathcal{I}\\ l\neq n}} \frac{1}{a_n - a_l} \qquad \forall n \in \mathcal{I} \subseteq \mathbb{Z}. \tag{1.4}$$

The solution to the IVP (1.1)–(1.2) is the meromorphic function

$$u_\nu(x,t) = \frac{x}{t} - 2\nu \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{1}{x - i\beta_n(t,\nu)} = \frac{x}{t} - 2\nu \sum_{n=1}^{\infty} \frac{2x}{x^2 + \beta_n^2(t,\nu)}, \tag{1.5}$$

where $\{\pm i\beta_n\}_{n\in\mathbb{Z}^*}$ is a countable set of pure imaginary conjugate poles (the zeros of the Cole–Hopf variable) satisfying $\beta_{-n} = -\beta_n$. The motion of these poles on the imaginary axis is governed by an infinite-dimensional CDS:

$$\dot{\beta}_n = \frac{\beta_n}{t} - 2\nu \sum_{\substack{l=-\infty\\ l\neq n,0}}^{\infty} \frac{1}{\beta_l - \beta_n} = \frac{\beta_n}{t} + \frac{\nu}{\beta_n} - 2\nu \sum_{\substack{l=1\\ l\neq n}}^{\infty} \frac{2\beta_n}{\beta_l^2 - \beta_n^2} \quad \forall n \in \mathbb{Z}^*. \tag{1.6}$$

Numerical simulations of the evolution of these poles and the solution for small viscosity are described in [18]. For more details on the derivation of (1.5) and (1.6), see the companion article [18, section 2]. As $\nu \to 0^+$, these poles condense on the imaginary axis for all $t > 0$. The asymptotic distance between two successive poles as $\nu \to 0^+$ is proportional to ν when the index of these poles grows like $k \sim 1/\nu$ (see Fig. 1.1). This condensation phenomenon is captured by an asymptotic density of poles (also referred to as the limiting pole density in the work of Bessis and Fournier

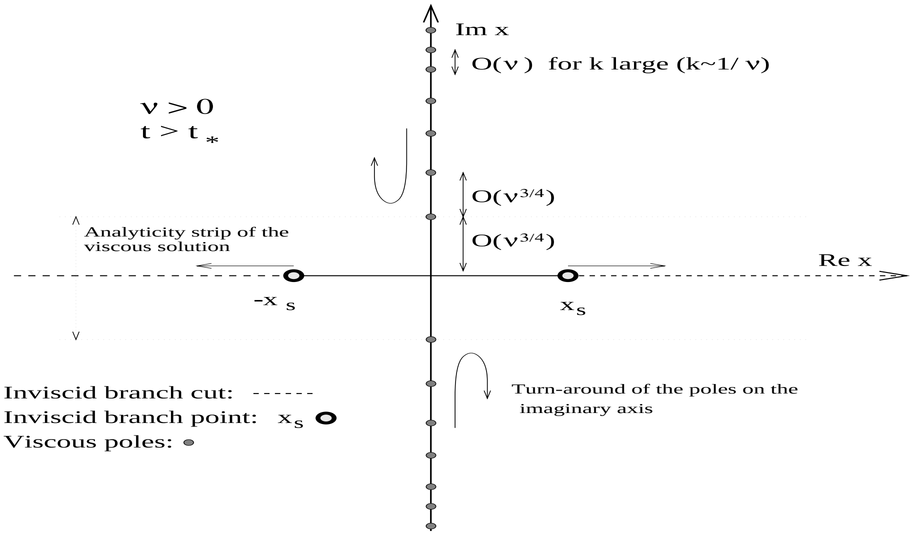

FIG. 1.2. *Inviscid branch points, branch cuts, and viscous poles for $\nu > 0$ and $t > t_*$. The inviscid branch points have coalesced at $t = t_*$ at the origin and are now moving away from each other on the real axis. However, the poles are fixed to the imaginary axis and are asymptotically given by $\beta_k(t,\nu) = \mathcal{O}(\nu^{3/4})$ as $\nu \to 0^+$ and $\Delta\beta_k = \mathcal{O}(\nu)$ as $\nu \to 0^+$ for k large ($k \sim 1/\nu$). They turn around and move away from the origin at a time $t_u > t_*$ if ν is small enough (if $\nu \gtrapprox .01$, $t_u < t_*$).*

[4]). We show that this density depends directly on the relevant saddle points of the small ν asymptotic expansion of the component $E_\nu(x,t)$ which carries the zeros of the Cole–Hopf variable. From the Cole–Hopf transformation, we find that

$$u_\nu(x,t) = \frac{x}{t} - 2\nu\,\partial_x \log\big(E_\nu(x,t)\big), \quad E_\nu(x,t) = \int_{-\infty}^{\infty} e^{w(z,x)/2\nu}\,dz,$$

where $w(z,x)$ is the phase function defined by

$$w(z,x) = \int_0^z \left(\frac{x}{t} - \frac{\eta}{t} - u_0(\eta)\right) d\eta.$$

The saddle points $z_s(\beta,t)$ of the phase function $w(z,x)$ are implicitly given by

$$0 = w_z(z_s, i\beta) = \frac{i\beta}{t} - \frac{z_s}{t} - u_0(z_s).$$

Let $\sigma(\beta;t)$ be a cumulative distribution function corresponding to the integral of the asymptotic density of poles (which we will define later). That is, $\sigma(\beta;t)$ counts the number of poles contained within the interval $[0,\beta]$. Then it is shown that

$$\sigma(\beta;t) = \frac{\Im w(z_s, i\beta)}{\pi},$$

and then the asymptotic pole locations are implicitly given by the equation

$$\frac{\sigma(\beta;t)}{2\nu} = k - 1/2, \qquad k \to +\infty.$$

We show that the asymptotic density of poles defined by Bessis and Fournier in [3, 4] as

$$\rho(\beta;t) = \lim_{\substack{n\to\infty \\ \nu\to 0^+}} \left.\frac{2\nu}{\Delta\beta_n(t,\nu)}\right|_{\beta_n=\beta}$$

is given by

$$\rho(\beta;t) = \frac{d}{d\beta}\sigma(\beta;t) = \frac{1}{\pi t}\,\Re z_s^+(\beta;t),$$

where $z_s^+(\beta;t)$ is the relevant saddle point with a positive real part in the expansion of $E_\nu(i\beta,t)$ as $\nu \to 0^+$. This density is explicitly calculated for all $t > 0$ using Cardan's formula. The Mittag–Leffler expansion (1.5) has an integral representation which is valid away from the imaginary axis. It can be expressed as the integral of a continuous function against the distributional derivative of a nonnegative regular finite Borel measure defined for $|\beta| \le \beta_{\max} < +\infty$ as

$$\sigma_\nu(\beta;\beta_{\max},t) = \int_{-\beta_{\max}}^{\beta} 2\nu \sum_{k=1}^{N_\nu} \left[\delta\left(\xi - \beta_k(t,\nu)\right) + \delta\left(\xi + \beta_k(t,\nu)\right)\right] d\xi,$$

where

$$N_\nu(\beta_{\max}) = \sup_{0<\delta\le t\le t_*} \left[\frac{\sigma(\beta_{\max};t)}{2\nu}\right] < +\infty. \tag{1.7}$$

The "polar" measure $\sigma_\nu(\beta;\beta_{\max},t)$ is analogous to the spectral measure in the KdV problem (cf. [12, 15]). The zero-viscosity limit of the pole expansion is found by taking the weak limit of $d\sigma_\nu(\beta;\beta_{\max},t)/d\beta$ which approximates the asymptotic density of poles. Thus, we show that for $\beta \in [-\beta_{\max},\beta_{\max}]$, $0 < \beta_{\max} < +\infty$,

$$\rho(\beta;t) \equiv \text{w-}\lim_{\nu\to 0^+} \frac{d\sigma_\nu}{d\beta}(\beta;\beta_{\max},t),$$

where w-$\lim_{\nu\to 0^+}$ denotes the weak limit of measures. The limiting integral representation of the solution is given by the nonparametric form

$$u(x,t) = \frac{x}{t} - \int_{-\infty}^{\infty} \frac{\rho(\beta;t)}{x - i\beta}\,d\beta = \frac{x}{t} - x\int_{-\infty}^{\infty} \frac{\rho(\beta;t)}{x^2+\beta^2}\,d\beta.$$

This representation is computed via residues, and the analytic structure of the inviscid solution is explicitly recovered up to t_*.

We also show that the continuum limit of the pair of equations consisting of the pole expansion (1.5) and the dynamical system (1.6) is the system of two integro-differential equations in parametric form:

$$\begin{aligned}\frac{\partial f}{\partial t}(\zeta,t) &= \frac{f(\zeta,t)}{t} - P.V.\int_{-\infty}^{\infty} \frac{d\zeta'}{f(\zeta,t) - f(\zeta',t)} \\ &= \frac{f(\zeta,t)}{t} - f(\zeta,t)\,P.V.\int_{-\infty}^{\infty} \frac{d\zeta'}{f^2(\zeta,t) - f^2(\zeta',t)}\end{aligned}$$

and

$$u(x,t) = \frac{x}{t} - \int_{-\infty}^{\infty} \frac{d\zeta'}{x - f(\zeta',t)} = \frac{x}{t} - x\int_{-\infty}^{\infty} \frac{d\zeta'}{x^2 - f^2(\zeta',t)},$$

in which the spatial branch cuts are defined by the condition $x \neq f(\zeta, t)$ for each fixed $t > 0$. This 2×2 system is shown to be equivalent to the characteristic equations of the inviscid BE.

The equivalence between the parametric and nonparametric form of the integral representation of the inviscid solution is obtained by introducing a simple change of variable $i\beta = f(\zeta, t)$ in the parametric equations. From this analysis, we clarify the relation between the pole positions and their asymptotic density. We show for $t = t_*$ that the pole positions can be recovered from the asymptotic density by choosing the right discretization on the "continuum" curve on which this density lies.

Furthermore, the analyticity properties of the inviscid solution can be analyzed by describing the asymptotic behavior of its Fourier transform (see [13, 20]). We find a uniform asymptotic expansion as $k \to +\infty$ of the Fourier transform of the inviscid solution, clarifying the seemingly discontinuous change of behavior of $\hat{u}(k,t)$ at t_* presented in [13]. This discontinuity in the asymptotic behavior is a direct consequence of the coalescence of the two second-order branch points $\pm x_s(t)$ into a third-order branch point at the origin $x_s(t_*) = 0$. We show that as $k \to +\infty$, $\hat{u}(k,t) = \mathcal{C}_0 \cdot (tk)^{-4/3}\mathrm{Ai}[(-3ikx_s(t)/2)^{2/3}]\left(1 + \mathcal{O}\left(k^{-1}\right)\right)$. From the (uniform) asymptotic expansion of the Airy function we find that $\hat{u}(k,t) \sim \mathcal{C}_1(t) \cdot (t_* - t)^{-1/4}\, k^{-3/2} \exp(-k|x_s(t)|)$ for $0 < t < t_*$ and $\hat{u}(k,t_*) \sim \mathcal{C}_2 \cdot (t_* k)^{-4/3}$, where $\mathcal{C}_0$, $\mathcal{C}_1(t)$, $\mathcal{C}_2$ are appropriate numerical constants.

2. Polar measure, integral representation, and inviscid limit. In [18], the solution to the IVP (1.1)–(1.2) is constructed in the following property.

PROPERTY 2.1. *Let $2\alpha = 1/t_* - 1/t$ for $\nu, t, t_* > 0$; then*

$$u_\nu(x,t) = \frac{x}{t} - 2\nu\, \partial_x \log\bigl(E_\nu(x,t)\bigr),$$
$$E_\nu(x,t) = \int_{-\infty}^{\infty} \exp\left\{ \frac{1}{2\nu} \left(\frac{x}{t} y + \alpha y^2 - y^4 \right) \right\} dy.$$

Furthermore, $u_\nu(x,t)$ has a Mittag–Leffler (pole) representation:

$$u_\nu(x,t) = \frac{x}{t} - 2\nu \sum_{n=1}^{\infty} \frac{2x}{x^2 + \beta_n^2(t,\nu)},$$

which converges uniformly on compact sets for x away from the poles $x = \pm i\beta_n$.

From the integral representation of the solution one can describe the behavior of the solution as the viscosity tends to zero: using a saddle-point analysis, we have shown in [18, section 4.2] that the dominant behavior of $E_\nu(i\beta, t)$ as $\nu \to 0^+$ or as $\beta \to +\infty$ is given by an asymptotic relation of the form

$$\sqrt{\frac{|6z_0(\beta;t)^2 - \alpha|}{2\pi\nu}}\ \exp\left\{ -\frac{1}{2\nu} \Re w\bigl(z_0(\beta;t), i\beta\bigr) \right\} E_\nu(i\beta,t)$$
$$= \cos\left(\frac{\Im w(z_0(\beta;t), i\beta)}{2\nu} - \frac{\theta(z_0(\beta;t),t)}{2} \right) + \mathcal{O}\left(\frac{\nu}{\beta^{4/3}} \right),$$

where

$$w(z,i\beta) = \int_0^z \left(i\beta/t - \eta/t - u_0(\eta)\right)\, d\eta = i\beta z/t + \alpha z^2 - z^4,$$
$$\theta(z,t) = \arg(\partial_z^2 w) = \arg(6z^2 - \alpha), \quad -\pi \leq \theta(z,t) < \pi,$$

and $z_s(\beta;t)$ is implicitly defined by

$$0 = w_z(z_s(\beta;t), i\beta) = \frac{i\beta}{4t} + \frac{\alpha}{2} z_s - z_s^3, \qquad s = 0, 1, 2. \tag{2.1}$$

Solving (2.1) using Cardan's formula (see Appendix B) and separating real and imaginary parts, we find that

$$\begin{cases} z_0 = \frac{\sqrt{3}}{2}(\mathcal{A} - \mathcal{B}) + \frac{i}{2}(\mathcal{A} + \mathcal{B}), \\ z_1 = -\overline{z_0} = \frac{\sqrt{3}}{2}(\mathcal{B} - \mathcal{A}) + \frac{i}{2}(\mathcal{A} + \mathcal{B}), \\ z_2 = -i(\mathcal{A} + \mathcal{B}), \end{cases} \tag{2.2}$$

where for $\beta > |x_s(t)|$,

$$\begin{cases} \mathcal{A}(\beta;t) = (8t)^{-1/3} \sqrt[3]{\beta + \sqrt{\beta^2 + x_s^2}} > 0, \\ \mathcal{B}(\beta;t) = (8t)^{-1/3} \sqrt[3]{\beta - \sqrt{\beta^2 + x_s^2}} > 0. \end{cases} \tag{2.3}$$

For $\beta < -|x_s(t)|$, $\mathcal{A}$ and $\mathcal{B}$ are defined by the odd parity condition $\mathcal{A}(-\beta;t) = -\mathcal{A}(\beta;t)$ and $\mathcal{B}(-\beta;t) = -\mathcal{B}(\beta;t)$ so that

$$z_s(-\beta;t) = -z_s(\beta;t). \tag{2.4}$$

For small ν (fixed k) or for large β (large k, fixed ν), the poles β_k are approximated by the roots of the equation

$$\frac{\sigma(\beta;t)}{2\nu} - \frac{1}{2\pi}\theta(z_0(\beta;t), t) = k - \frac{1}{2}, \quad k \in \mathbb{N}^*, \tag{2.5}$$

with the convention that $\beta_{-k} \equiv -\beta_k$. Clearly, since $|\theta| \le \pi$, the contribution of θ is negligible compared with that of σ. Thus, we approximate (2.5) by

$$\frac{\sigma(\beta;t)}{2\nu} \approx k - \frac{1}{2}, \quad k \in \mathbb{N}^*. \tag{2.6}$$

Choose a parameter $\beta_{\max} < +\infty$, and then choose $N_\nu(\beta_{\max})$ as follows: let $\mathrm{Int}[x]$ denote the integer part of x with half-integers rounded down, and fix $\nu > 0$. Then for any $\delta > 0$ and compact set $[\delta, t_*]$, define N_ν by

$$N_\nu(\beta_{\max}) = \sup_{0<\delta\le t\le t_*} \left[\frac{\sigma(\beta_{\max};t)}{2\nu}\right] < +\infty. \tag{2.7}$$

The $\beta_k(t,\nu)$ are ordered as follows:

$$0 \le |x_s(t)| < \beta_1(t,\nu) < \cdots < \beta_k(t,\nu) < \cdots < \beta_{N_\nu} < +\infty \tag{2.8}$$

for $1 < k < N_\nu$. For negative indices, the ordering of the β_{-k}'s is the reverse of that given in (2.8).

Let $U_\nu^{\beta_{\max}}(x,t)$ be the N_νth partial sum of $U_\nu(x,t)$:

$$\begin{aligned} U_\nu^{\beta_{\max}}(x,t) = x - t\, u_\nu^{\beta_{\max}}(x,t) &= t \cdot 2x \cdot \sum_{n=1}^{N_\nu} \frac{2\nu}{x^2 + \beta_n^2} \\ &= t \cdot 2\nu \sum_{n=1}^{N_\nu} \left(\frac{1}{x - i\beta_n} + \frac{1}{x + i\beta_n}\right). \end{aligned} \tag{2.9}$$

Let $U_\nu(x,t)$ be the spatially singular part of the viscous solution defined by

$$U_\nu(x,t) = x - t\,u_\nu(x,t) = t \cdot 2x \sum_{n=1}^{\infty} \frac{2\nu}{x^2+\beta_n^2(t,\nu)}, \tag{2.10}$$

and let the remainder $R_\nu^{\beta_{\max}}(x,t)$ be defined by

$$R_\nu^{\beta_{\max}}(x,t) = U_\nu(x,t) - U_\nu^{\beta_{\max}}(x,t) = t \cdot 2x \sum_{n=N_\nu+1}^{\infty} \frac{2\nu}{x^2+\beta_n^2(t,\nu)}.$$

Let $\delta(\beta)$ denote the usual Dirac measure, and define the density $\sigma_\nu(\beta;\beta_{\max},t)$ with support in $[-\beta_{\max},\beta_{\max}]$ by

$$\sigma_\nu(\beta;\beta_{\max},t) = \int_{-\beta_{\max}}^{\beta} 2\nu \sum_{k=1}^{N_\nu} \left[\delta\left(\xi-\beta_k(t,\nu)\right)+\delta\left(\xi+\beta_k(t,\nu)\right)\right] d\xi. \tag{2.11}$$

Since the poles β_n are ordered according to (2.8), this insures that (2.11) is nonnegative and vanishes outside $[-\beta_{\max},\beta_{\max}]$. Moreover, once $\beta_{\max}$ has been chosen, $\sigma_\nu(\beta;\beta_{\max},t)$ is uniformly bounded for $0<\delta\le t\le t_*$ by $2\nu N_\nu$. Thus, from the definition (2.7) of N_ν, $\sigma_\nu(\beta;\beta_{\max},t)$ is uniformly bounded in ν, and thus is a regular finite Borel measure. Since

$$\begin{aligned} d\sigma_\nu(\beta;\beta_{\max},t) &= 2\nu \sum_{k=1}^{N_\nu} \left[\delta\left(\beta-\beta_k(t,\nu)\right)+\delta\left(\beta+\beta_k(t,\nu)\right)\right] d\beta \\ &= 2\nu \sum_{\substack{k=-N_\nu \\ k\neq 0}}^{N_\nu} \delta\left(\beta-\beta_k(t,\nu)\right) d\beta, \end{aligned} \tag{2.12}$$

the measure $d\sigma_\nu(\beta;\beta_{\max},t)$ consists of a sum of delta functions with weight (height) 2ν decreasing as $\nu\to 0^+$. The density of these delta functions increases like $1/\Delta\beta_k = \mathcal{O}(1/\nu)$ as $\nu\to 0^+$ for $k\sim 1/\nu$. Since the measure $d\sigma_\nu$ is odd ($d\sigma_\nu(-\beta;\beta_{\max},t) = -d\sigma_\nu(\beta;\beta_{\max},t)$), we can represent $U_\nu^{\beta_{\max}}(x,t)$ as

$$U_\nu^{\beta_{\max}}(x,t) = t\cdot \int_{-\infty}^{\infty} \frac{d\sigma_\nu(\beta;\beta_{\max},t)}{x-i\beta} = t\cdot 2x \int_0^{\infty} \frac{d\sigma_\nu(\beta;\beta_{\max},t)}{x^2+\beta^2}, \tag{2.13}$$

where the last integral should be understood as

$$\int_0^{\infty} \frac{d\sigma_\nu(\beta;\beta_{\max},t)}{x^2+\beta^2} \equiv \frac{1}{2}\int_{-\infty}^{\infty} \frac{d\sigma_\nu(\beta;\beta_{\max},t)}{x^2+\beta^2}. \tag{2.14}$$

Since the measure $d\sigma_\nu(\beta;\beta_{\max},t)$ has support in the compact interval $[-\beta_{\max},\beta_{\max}]$, we have proved the following property.

PROPERTY 2.2. *$U_\nu^{\beta_{\max}}(x,t)$ has an integral representation for $x\notin[-i\beta_{\max},i\beta_{\max}]$ given by*

$$U_\nu^{\beta_{\max}}(x,t) = t\cdot \int_{-\beta_{\max}}^{\beta_{\max}} \frac{d\sigma_\nu(\beta;\beta_{\max},t)}{x-i\beta} = t\cdot x \int_{-\beta_{\max}}^{\beta_{\max}} \frac{d\sigma_\nu(\beta;\beta_{\max},t)}{x^2+\beta^2}.$$

Now that we have established the validity of the integral representation of Property 2.2, we use this to derive an integral representation of the inviscid solution: we introduce the asymptotic pole density $\rho(\beta; t)$, which also corresponds to the asymptotic distribution of the zeros of E_ν. We define it in relation to the limiting measure $\sigma(\beta; t)$ as follows.

DEFINITION 2.1. *For $\beta \in [-\beta_{\max}, \beta_{\max}]$, the cumulative distribution function $\sigma(\beta; t)$ represents the number of poles within the interval $[0, \beta] \subset [0, \beta_{\max}]$.*

$$\sigma(\beta; t) = \int_0^\beta \rho(\zeta; t)\, d\zeta = \frac{1}{\pi} \Im w(z_s(\beta; t), i\beta).$$

It follows directly from the definition (2.5) of the zeros $\pm i\beta_k(t, \nu)$ of $E_\nu(x, t)$ and from Definition 2.1 of the asymptotic density of the zeros that

$$\lim_{\nu \to 0^+} \sum_{k=1}^{N_\nu} \frac{2\nu}{x^2 + \beta_k^2(t, \nu)} = \int_{-\beta_{\max}}^{\beta_{\max}} \frac{\rho(\beta; t)}{x^2 + \beta^2}\, d\beta = \frac{U^{\beta_{\max}}(x, t)}{tx} \tag{2.15}$$

for a fixed $\delta > 0$ and $0 < \delta \leq t \leq t_*$. Thus, we have that

$$|U^{\beta_{\max}}(x, t) - U_\nu^{\beta_{\max}}(x, t)| = t\, |x| \cdot \left| \int_{-\beta_{\max}}^{\beta_{\max}} \frac{d\sigma - d\sigma_\nu}{x^2 + \beta^2} \right| < \epsilon/3 \tag{2.16}$$

for ν small enough on compact sets for x and t away from the branch cuts defined by $(-i\infty, -i|x_s|] \cup [i|x_s|, +i\infty)$ (for a similar argument see, for example, [12]). Thus the convergence of the measure $d\sigma_\nu$ to $d\sigma$ is described in the following way.

PROPERTY 2.3. *For $\beta \in [-\beta_{\max}, \beta_{\max}]$ and $0 < \delta \leq t \leq t_*$, the sequence of distributions $d\sigma_\nu(\beta; \beta_{\max}, t)$ converges weakly to $d\sigma(\beta; t)$:*

$$\text{w-}\lim_{\nu \to 0^+} d\sigma_\nu(\beta; \beta_{\max}, t) = d\sigma(\beta; t) = \rho(\beta; t)\, d\beta = \frac{1}{\pi} \Im dw(z_s(\beta; t), i\beta).$$

This measure is analogous to the spectral measure introduced in [12, 15]. Here $\text{w-lim}_{\nu \to 0^+}$ stands for a limit in the sense of weak convergence of measures; that is, $\text{w-lim}_{\nu \to 0^+} d\mu_\nu(\beta) = d\mu(\beta)$ if

$$\lim_{\nu \to 0^+} (\phi, d\mu_\nu) = (\phi, d\mu) = \int_{-\beta_{\max}}^{\beta_{\max}} \phi(\beta)\, d\mu(\beta)$$

for every continuous function ϕ in $[-\beta_{\max}, \beta_{\max}]$. Note that we suspect the convergence

$$\lim_{\nu \to 0^+} U_\nu^{\beta_{\max}}(x, t) = U^{\beta_{\max}}(x, t) \tag{2.17}$$

to hold uniformly over compact sets for t and x away from the branch cuts.

From the definition of the limiting function $U(x, t)$

$$U(x, t) = t \cdot x \int_{-\infty}^{\infty} \frac{\rho(\beta; t)}{x^2 + \beta^2}\, d\beta,$$

the remainder $R^{\beta_{\max}}(x, t)$, defined for $x \notin (-i\infty, -i\beta_{\max}] \cup [i\beta_{\max}, i\infty)$ as

$$R^{\beta_{\max}}(x, t) = U^{\beta_{\max}}(x, t) - U(x, t) = t \cdot 2x \cdot \int_{|\beta| \geq \beta_{\max}} \frac{\rho(\beta; t)}{x^2 + \beta^2}\, d\beta, \tag{2.18}$$

can be shown to go to zero as $\beta_{\max} \to +\infty$ independently of ν: fix an $R > 0$ such that $|x| \leq R < \beta_{\max}$, then $|x^2 + \beta^2| \geq \beta^2 - R^2$. Let $\theta > 1$ be a fixed parameter; then for $\beta > \beta_{\max} > \sqrt{\theta/(\theta-1)}R$, we have $1/(\beta^2 - R^2) < \theta/\beta^2$. Then since $\rho(\beta;t) = \mathcal{O}(\beta^{1/3})$ as $\beta \to +\infty$ (see Theorem 3.1), we can estimate (2.18) as $\beta_{\max} \to +\infty$ as follows:

$$\left| \int_{|\beta| \geq \beta_{\max}} \frac{\rho(\beta;t)}{x^2 + \beta^2}\, d\beta \right| \leq \mathcal{C}(t) \int_{|\beta| \geq \beta_{\max}} \frac{\beta^{1/3}}{\beta^2 - R^2}\, d\beta$$
$$\leq \theta \cdot \mathcal{C}(t) \cdot \int_{|\beta| \geq \beta_{\max}} \frac{d\beta}{\beta^{5/3}} = \mathcal{O}(\beta_{\max}^{-2/3}).$$

Therefore, on compact sets for x and t,

$$|U^{\beta_{\max}}(x,t) - U(x,t)| < \epsilon/3 \tag{2.19}$$

for $\beta_{\max}$ large enough independent of ν.

The last estimate concerns

$$\begin{aligned} |R_\nu^{\beta_{\max}}(x,t)| &= |U_\nu^{\beta_{\max}}(x,t) - U_\nu(x,t)| \\ &= t\,|2x| \left| \sum_{n=N_\nu+1}^{\infty} \frac{2\nu}{x^2 + \beta_n^2(t,\nu)} \right|. \end{aligned} \tag{2.20}$$

In [18, section 4.2], it is shown that $\beta_n(t,\nu) = \mathcal{O}\left((n\nu)^{3/4}\right)$ as $n \to +\infty$ for fixed $\nu > 0$. Therefore, let $y_n(t,\nu) = \mathcal{C}n\nu$, where $\mathcal{C}$ is an appropriate asymptotic constant which depends on t (see (5.1) for such a representation). This assumption also can be justified by combining (5.1) and the fact that the order $\lambda = 4/3$ of the entire function E_ν is also the order of convergence of its zeros (see [18, section 2.1]). Then since $N_\nu(\beta_{\max}) = \sup_{0<\delta\leq t\leq t_*} \mathrm{Int}[\sigma(\beta_{\max};t)/2\nu]$, following a similar argument as in the proof of (2.19), we may estimate (2.20) as

$$|U_\nu^{\beta_{\max}}(x,t) - U_\nu(x,t)| \leq t\,|2x| \left| \sum_{n=N_\nu+1}^{\infty} \frac{2\nu}{x^2 + y_n^{3/2}} \right|$$
$$\leq t\,|2x|\,\mathcal{C}_1 \left| \int_{\beta_{\max}}^{+\infty} \frac{dy}{x^2 + y^{3/2}} \right| = \mathcal{O}(\beta_{\max}^{-1/2})$$

as $\beta_{\max} \to +\infty$, uniform in ν on compact sets for $t \in [\delta, t_*]$ and x away from the branch cuts. Choosing $\beta_{\max}$ large enough so that $|U(x,t) - U^{\beta_{\max}}(x,t)| < \epsilon/3$ and $|U_\nu^{\beta_{\max}}(x,t) - U_\nu(x,t)| < \epsilon/3$ independent of ν and then choosing ν small enough in such a way that $|U^{\beta_{\max}}(x,t) - U_\nu^{\beta_{\max}}(x,t)| < \epsilon/3$, we finally have

$$|U_\nu(x,t) - U(x,t)| \leq |U(x,t) - U^{\beta_{\max}}(x,t)|$$
$$+ |U^{\beta_{\max}}(x,t) - U_\nu^{\beta_{\max}}(x,t)| + |U_\nu^{\beta_{\max}}(x,t) - U_\nu(x,t)| < \epsilon.$$

Using the fact that $\rho(\beta;t) = 0$ for $|x| < |x_s(t)|$ when $0 < t \leq t_*$, we have proved the following theorem.

THEOREM 2.4. *For $\delta > 0$, $t \in [\delta, t_*]$; on compact sets for x away from the branch cuts defined by $(-i\infty, -i|x_s|] \cup [i|x_s|, +i\infty)$, we have*

$$\lim_{\nu \to 0^+} U_\nu(x,t) = t \cdot 2x \int_{|x_s(t)|}^{\infty} \frac{\rho(\beta;t)}{x^2 + \beta^2}\, d\beta.$$

3. Asymptotic density of poles. Now that we have defined the asymptotic density of poles, we proceed with its explicit computation for the different time intervals $(0, t_*)$, $t = t_*$ and $(t_*, +\infty)$. As in section 2, let w-$\lim_{\nu\to 0^+}$ denote a weak limit in the sense of weak convergence of measures. Then we prove the following.

THEOREM 3.1. it For $0 < \beta_{\max} < +\infty$, *the asymptotic density of poles* $\rho(\beta;t)$: $[-\beta_{\max}, \beta_{\max}] \times \mathbb{R}_+ \to \mathbb{R}_+$ *is a positive even function of* β *defined by*

$$\rho(\beta;t) \equiv \text{w-}\lim_{\nu\to 0^+} \frac{d\sigma_\nu}{d\beta}(\beta;\beta_{\max},t) = \frac{d\sigma}{d\beta}(\beta;t) = \frac{1}{\pi t}\,\Re z_s^+(\beta;t),$$

where $z_s^+(\beta;t)$ *is the saddle point with positive real part which is relevant to the asymptotic expansion of* $E_\nu(i\beta,t)$ *as* $\nu \to 0^+$. *This saddle point is determined by the implicit equation*

$$\frac{\partial w}{\partial z}\big(z_s(\beta;t), i\beta\big) = \frac{i\beta}{t} - \frac{z_s(\beta;t)}{t} - u_0\big(z_s(\beta;t)\big) = 0.$$

Let $\pm x_s(t) = \pm i\,(3t_*)^{-3/2}(t_* - t)^{3/2}t^{-1/2}$ *be the second-order branch points of the inviscid solution arising from the initial data* $u_0(x) = 4x^3 - x/t_*$. *For* $t > t_*$ *and for* $t < t_*$, $|\beta| > |x_s|$,

$$\rho(\beta;t) = \frac{2^{2/3}\sqrt{3}}{\pi}(4t)^{-4/3}\left\{\sqrt[3]{|\beta| + \sqrt{\beta^2 + x_s^2}} - \sqrt[3]{|\beta| - \sqrt{\beta^2 + x_s^2}}\right\}.$$

For $t < t_*$, $|\beta| < |x_s|$,

$$\rho(\beta;t) = 0.$$

For $t = t_*$,

$$\rho(\beta;t_*) = \frac{2\sqrt{3}}{\pi}(4t_*)^{-4/3}|\beta|^{1/3}.$$

For $t > t_*$, $\beta = 0$,

$$\rho(0;t) = \lim_{\substack{\beta\to 0\\ t>t_*}} \rho(\beta;t) = \frac{1}{2\pi}(t - t_*)^{1/2}t^{-3/2}t_*^{-1/2}.$$

Proof. In [3, 4], Bessis and Fournier introduced a limiting density of poles which characterizes the process of condensation of poles on the imaginary axis as the viscosity $\nu \to 0^+$. They defined it in [3] in the following way:

$$\rho(\beta;t) \equiv \lim_{\substack{n\to\infty\\ \nu\to 0^+}} \left.\frac{2\nu}{\Delta\beta_n(t,\nu)}\right|_{\beta_n=\beta} : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}_+,$$

with $\Delta\beta_n(t,\nu) = \beta_{n+1}(t,\nu) - \beta_n(t,\nu) > 0$, where $n \in \mathbb{Z}\backslash\{0\} = \{\pm 1, \pm 2, \dots\}$, with the convention that $\beta_{-n} = -\beta_n$. Since

$$\frac{\partial w}{\partial \beta}\big(z_s(\beta;t), i\beta\big) = \frac{i}{t}\,z_s(\beta;t), \tag{3.1a}$$

$$\frac{\partial w}{\partial z}\big(z_s(\beta;t), i\beta\big) = \frac{i\beta}{t} - \frac{z_s(\beta;t)}{t} - u_0\big(z_s(\beta;t)\big) = 0. \tag{3.1b}$$

From Property 2.3 we find that the density is given by

$$\rho(\beta;t) = \frac{1}{\pi}\Im\frac{dw}{d\beta}\big(z_s(\beta;t), i\beta\big) = \frac{1}{\pi t}\,\Re z_s^+(\beta;t), \tag{3.2}$$

where $z_s^+(\beta;t)$ is the relevant saddle point with positive real part. Since $z_s(-\beta;t) = -z_s(\beta;t)$, in order to have $\rho(-\beta;t) = \rho(\beta;t) > 0$, we must take in both cases ($\beta > 0$ and $\beta < 0$) the saddle point with positive real part. That is, we must take z_0 for $\beta > 0$ and z_1 for $\beta < 0$ since they are related by $z_0(-\beta;t) = -z_0(\beta;t) = \overline{z_1}(\beta;t)$ and $z_1(-\beta;t) = -z_1(\beta;t) = \overline{z_0}(\beta;t)$ (see [18, section 4.2]). With this choice of saddle points, the asymptotic density defined in (3.2) is positive whether $\beta > 0$ or $\beta < 0$. Note that if we let $x = i\beta$, $x_0 = x_0(x,t) = z_s(\beta;t)$, then the inviscid solution is $u(x,t) = u_0(x_0(x,t),t)$, where $x - x_0 - t\,u_0(x_0) = 0$ (see Appendix C). Combining (2.2) and (3.2), we immediately have an expression for the density as a function of β and t:

$$\rho(\beta;t) = \frac{\sqrt{3}}{2\pi t}(\mathcal{A}(\beta;t) - \mathcal{B}(\beta;t)), \tag{3.3}$$

with $\mathcal{A}(\beta;t)$ and $\mathcal{B}(\beta;t)$ defined in (2.3).

We can now describe the various cases $t = t_*$, $0 < t < t_*$, $t > t_*$, and $x = 0, t > t_*$:

(i) $t = t_*$:

$$\rho(\beta;t_*) = \frac{1}{\pi t_*}\Re z_s(\beta;t_*) = \frac{\sqrt{3}}{2\pi t_*}\left(\frac{\beta}{4t_*}\right)^{1/3} = \frac{2\sqrt{3}}{\pi}(4t_*)^{-4/3}|\beta|^{1/3}. \tag{3.4}$$

In the last step of (3.4), we replace $\beta^{1/3}$ by $|\beta|^{1/3}$ to allow for both $\beta > 0$ and $\beta < 0$. Note that we can obtain (3.4) by taking the limit as $t \to t_*$ in (3.7) or (3.8). It is interesting to see that this density is the only one which explicitly can be computed from the formula introduced by Bessis and Fournier:

$$\rho(\beta;t) = \lim_{\substack{n\to\infty\\ \nu\to 0^+}} \left.\frac{2\nu}{\Delta\beta_n(t,\nu)}\right|_{\beta_n=\beta}. \tag{3.5}$$

The explicit pole positions $\beta_k(t_*,\nu)$ are given in part I by

$$\beta_k(t_*,\nu) = 4t_*\left(\frac{2\nu}{3\sqrt{3}}\right)^{3/4}\cdot\Big((k-1/3)^{3/4} + \mathcal{O}(1/k^{3/4})\Big) \tag{3.6}$$

as $k \to +\infty$ for all ν. Combining (3.5) with (3.6), one recovers (3.4).

(ii) $0 < t < t_*$: the density is zero for $|\beta| \le |x_s|$ and for $|\beta| > |x_s|$;

$$\rho(\beta;t) = \frac{2^{2/3}\sqrt{3}}{\pi}(4t)^{-4/3}\left\{\sqrt[3]{|\beta| + \sqrt{\beta^2 + x_s^2}} - \sqrt[3]{|\beta| - \sqrt{\beta^2 + x_s^2}}\right\}. \tag{3.7}$$

The behavior of $\rho(\beta;t)$ in a neighborhood of $\beta = \pm|x_s|$, ($|\beta| > |x_s|$) is

$$\rho(\beta;t) = \frac{t^{-4/3}}{\sqrt{6}\pi}\frac{\sqrt{|\beta| - |x_s|}}{|x_s|^{1/6}} + \mathcal{O}\Big(\big(|\beta| - |x_s|\big)^{3/2}\Big),$$

as mentioned in [3].

(iii) $t > t_*$: $\forall\beta \in \mathbb{R}$,

$$\rho(\beta;t) = \frac{2^{2/3}\sqrt{3}}{\pi}(4t)^{-4/3}\left\{\sqrt[3]{\beta+\sqrt{\beta^2+x_s^2}}+\sqrt[3]{-\beta+\sqrt{\beta^2+x_s^2}}\right\}. \tag{3.8}$$

(iv) $x = 0, t \geq t_*$: an interesting case occurs at the origin for $t > t_*$, as was pointed out by Bessis and Fournier in [3, 4]. The inviscid solution $u(0,t)$ at the shock is given by the asymptotic density of poles $\rho(0;t)$ (see (3.12)). If we look at the solution at the origin ($\beta \to 0$), $z_s(0;t)$ is the solution to

$$-\frac{z_s(0;t)}{t} = u_0(z_s(0;t)).$$

When $u_0(x) = 4x^3 - x/t_*$,

$$-\frac{z_s(0;t)}{t} = 4z_s(0;t)^3 - \frac{z_s(0;t)}{t_*}.$$

The nonzero pair of opposite saddle points are, therefore,

$$z_s^{\pm}(0;t) = \pm\frac{1}{2}\sqrt{\frac{t-t_*}{tt_*}} = \pm\sqrt{\frac{\alpha}{2}} \geq 0 \qquad \text{when} \;\; t \geq t_*.$$

The corresponding density is easily found to be

$$\rho(0;t) = \frac{1}{\pi t}\,\Re z_s^+(0;t) = \begin{cases} \frac{1}{2\pi}(t-t_*)^{1/2}t^{-3/2}t_*^{-1/2} & t > t_*, \\ 0 & t \leq t_*. \end{cases} \tag{3.9}$$

This could have been found by letting $\beta \to 0$ in (3.8). Moreover, it makes sense that the density $\rho(0;t)$ is null when $t < t_*$ since all the poles β_n are located above the inviscid branch points x_s on the imaginary axis, and $|x_s| > 0$.

3.1. Residue computation of the integral representation of the inviscid limit for $t = t_*$. Since $x_s(t_*) = 0$, we have

$$\int_0^\infty \frac{z^{1/3}}{x^2+z^2}\,dz = \frac{\pi}{\sqrt{3}\,x^{2/3}}. \tag{3.10}$$

Combining (3.4) and (3.10), we recover the inviscid solution at $t = t_*$:

$$u(x,t_*) = \lim_{\nu\to 0^+} u_\nu(x,t_*) = \frac{x}{t_*} - \left(\frac{x}{4t_*^4}\right)^{1/3} = \frac{x}{t_*} - \frac{U(x,t_*)}{t_*}. \tag{3.11}$$

For $0 < t < t_*$, a similar computation can be done using the double keyhole contour of integration displayed in Fig. 3.1.

For $t \geq t_*$, there is an interesting special case: since $z_s(0;t) \in \mathbb{R}$ for $t \geq t_*$, $\rho(0;t) = z_s(0;t)/\pi t$. Thus the inviscid solution at the origin (shock) for $t > t_*$ is given by (see (4.4b))

$$u(0,t) = \frac{u(0^-,t)-u(0^+,t)}{2} = \pi\rho(0;t) = \frac{1}{2}\,(t-t_*)^{1/2}t^{-3/2}\,t_*^{-1/2}, \tag{3.12}$$

i.e., the solution at the shock satisfies the jump condition (see (C.6), (4.4b), Fig. C.1, and [3, 16]):

$$u(0^{\mp},t) = \pm\frac{1}{2}\,(t-t_*)^{1/2}t^{-3/2}t_*^{-1/2}. \tag{3.13}$$

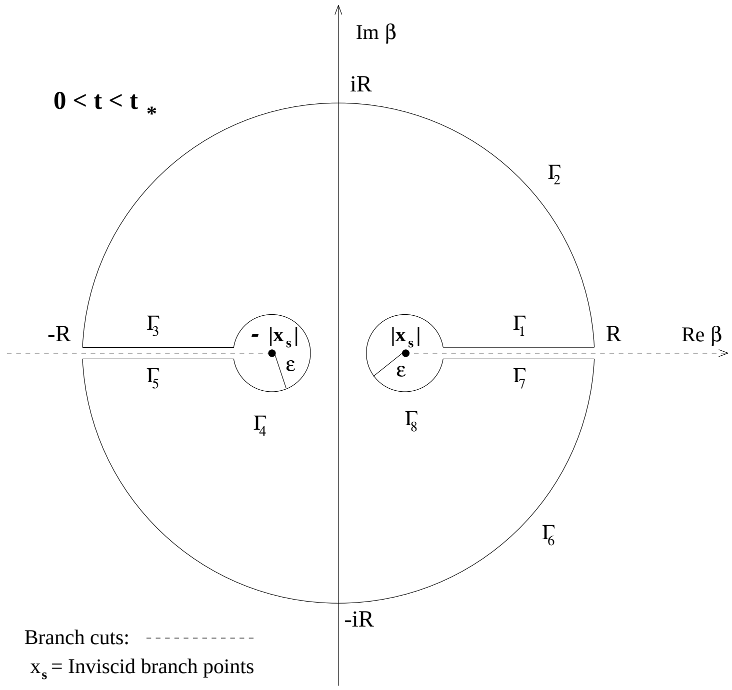

FIG. 3.1. *Contour of integration for the inviscid limit for $t < t_*$.*

Although this result was stated by Bessis and Fournier and can be derived by taking $\lim_{\beta\to 0}\rho(\beta;t)$, it is mentioned here to verify the validity formula $\rho(\beta;t) = (\pi t)^{-1}\,\Re z_s^+(\beta;t)$, where $z_s^+(\beta;t)$ is the saddle point relevant to the asymptotic expansion with a positive real part.

As a final remark, we would like to point out that this procedure which consists in recovering the analytic structure of the inviscid solution via the limiting pole density and the pole expansion is no longer possible when $t > t_*$ and $x \neq 0$. Indeed, in this case we are faced with the same (apparent) paradox that is present in the asymptotic expansion of the (spatial) Fourier transform of the inviscid solution (see section 6 and also [3]). Thus the only way to recover the inviscid solution for $t > t_*$ and $x \neq 0$ using the limiting pole density is by extending the solution obtained for $t < t_*$ to $t > t_*$.

4. Analytic extension of the integral representation of the inviscid solution on the imaginary axis. Let

$$u(x,t) = \frac{x}{t} - 2x\int_0^\infty \frac{\rho(\beta;t)}{x^2+\beta^2}\,d\beta = \frac{x}{t} - \int_{-\infty}^\infty \frac{\rho(\beta;t)}{x-i\beta}\,d\beta, \tag{4.1a}$$

$$\tilde{u}(y,t) = \frac{y}{t} + 2y\,P.V.\int_0^\infty \frac{\rho(\beta;t)}{y^2-\beta^2}\,d\beta = \frac{y}{t} + P.V.\int_{-\infty}^\infty \frac{\rho(\beta;t)}{y-\beta}\,d\beta. \tag{4.1b}$$

Then one can show that $\rho(y;t)$ is a density function which satisfies the conservation equation

$$\rho_t + (\rho\,\tilde{u})_y = 0. \tag{4.2}$$

Indeed, one only needs to verify that $u(x,t)$ defined by (4.1a) satisfies the inviscid BE $u_t + uu_x = 0$ under the assumption that (4.2) holds. Since u has branch cuts on

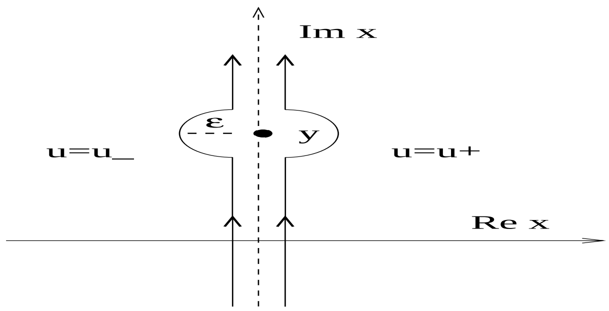

FIG. 4.1. *Analytic continuation of the integral representation of the inviscid solution on the imaginary axis.*

the imaginary axis for $t < t_*$, $\tilde{u}$ has branch cuts for y real (i.e., also on the imaginary axis); one can analytically continue u on the imaginary axis using the paths displayed in Fig. 4.1. Define the solution on the left (u_-) and right (u_+) of the imaginary axis by

$$u_\pm(iy,t) = \lim_{x\to 0^\pm} u(z = x+iy,t). \tag{4.3}$$

The discontinuity at $x = 0$ characterizes the shock solution. Thus we find that

$$\begin{aligned} u_+(iy,t) &= \frac{iy}{t} - P.V.\int_{-\infty}^{\infty} \frac{\rho(\beta;t)}{iy-i\beta}\,d\beta - \frac{\rho(y;t)}{i}\overbrace{\left(-\frac{1}{2}\oint_{|y-\beta|=\epsilon}\frac{d\beta}{y-\beta}\right)}^{\pi i} \\ &= i\left\{\frac{y}{t} + \int_{-\infty}^{\infty}\frac{\rho(\beta;t)}{y-\beta}\,d\beta + i\pi\rho(y;t)\right\}. \end{aligned}$$

Similarly, we have that

$$u_-(iy,t) = i\left\{\frac{y}{t} + \int_{-\infty}^{\infty}\frac{\rho(\beta;t)}{y-\beta}\,d\beta - i\pi\rho(y;t)\right\}.$$

Therefore,

$$\tilde{u}(y,t) = \frac{1}{2i}(u_+(iy,t) + u_-(iy,t)), \tag{4.4a}$$

$$\rho(y;t) = \frac{1}{2\pi}(u_-(iy,t) - u_+(iy,t)). \tag{4.4b}$$

Since $u_\pm$ are real on the real axis, it is clear that they satisfy the symmetry relations

$$\begin{aligned} u_\pm(\overline{x},t) &= \overline{u_\pm(x,t)}, \\ u_-(-x,t) &= -u_+(x,t), \end{aligned}$$

and therefore

$$u_\pm(iy,t) = \overline{u_\pm(-iy,t)} = -\overline{u_\mp(iy,t)}.$$

From this we have

$$\tilde{u}(y,t) = \Im u_+(iy,t), \tag{4.5a}$$

$$\rho(y;t) = -\frac{1}{\pi}\Re u_+(iy,t) = \frac{1}{\pi}\Re u_-(iy,t), \tag{4.5b}$$

and the symmetry relations

$$\tilde{u}(-y,t) = -\tilde{u}(y,t),$$
$$\rho(-y;t) = \rho(y;t).$$

Thus we have the following property.

PROPERTY 4.1.

$$u_{\pm}(iy,t) = \mp\pi\rho(y;t) + i\,\tilde{u}(y,t). \tag{4.6}$$

5. Continuum limit of the pole expansion and the CDS. Let $a_n(t,\nu) = i\beta_n(t,\nu)$, and define the complex map $\mathcal{F}(\zeta,\nu,t)$ as

$$a_n(t,\nu) = \mathcal{F}(\zeta_n^\nu = \nu n, \nu, t) : \mathbb{Z}^* \times \mathbb{R}_+^2 \to i\,\mathbb{R}_+, \quad a_{-n} = -a_n. \tag{5.1}$$

Reformulating the Mittag–Leffler expansion of u_ν (1.5) and the CDS (1.6) in terms of a_n, we have

$$u_\nu(x,t) = \frac{x}{t} - 2\nu \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \frac{1}{x-a_l}, \tag{5.2a}$$

$$\dot{a}_n = \frac{a_n}{t} - 2\nu \sum_{\substack{l=-\infty \\ l\neq n,0}}^{\infty} \frac{1}{a_n - a_l} \quad \forall n \in \mathbb{Z}^*. \tag{5.2b}$$

Both symmetric sums (5.2a, b) should be understood as

$$\sum_{\substack{l=-\infty \\ l\neq n,0}}^{\infty} \frac{1}{a_n-a_l} = \frac{1}{2a_n} + 2a_n \sum_{\substack{l=1 \\ l\neq n}}^{\infty} \frac{1}{a_n^2 - a_l^2} = \frac{1}{2a_n} + a_n \sum_{\substack{l=-\infty \\ l\neq 0,\pm n}}^{\infty} \frac{1}{a_n^2-a_l^2}$$

and

$$\sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \frac{1}{x-a_l} = 2x \sum_{l=1}^{\infty} \frac{1}{x^2-a_l^2} = x \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \frac{1}{x^2-a_l^2}.$$

At t_*, we have (cf. [18, section 4.1])

$$\begin{aligned} a_n(t_*,\nu) &= \mathcal{F}(\zeta_n^\nu = \nu n, \nu, t_*) = i\cdot 4t_*\,(2\nu\mu_n)^{3/4} \\ &= i\cdot 4t_*\,(2\nu(c_{-1}n + c_0 + c_1/n + \cdots))^{3/4} \\ &= i\cdot 4t_*\left(c_{-1}(2\nu n) + c_0 2\nu + c_1(2\nu)^2/(2\nu n) + \cdots\right)^{3/4}. \end{aligned}$$

Introduce the map

$$f(\zeta,t) = \mathcal{F}(\zeta,0,t) : \mathbb{R}\times\mathbb{R}_+ \to i\,\mathbb{R}_+, \qquad f(-\zeta,t) = -f(\zeta,t), \tag{5.3}$$

where the continuous variable ζ corresponds to a position on the real axis which can be thought of as a variable obtained by simultaneously letting $\nu\to 0^+$ and $n\to+\infty$. Assume that

$$a_n(t,\nu) = \mathcal{F}(n\nu,\nu,t) = f(n\nu,t) + e_n(\nu,t), \tag{5.4}$$

in which $e_n(\nu, t)$ is a small error term that goes to 0 as $\nu \to 0^+$. Thus, formally we have

$$2\nu \sum_{\ell \neq n} \frac{1}{a_n(t,\nu) - a_\ell(t,\nu)} \simeq 2\nu \sum_{\ell \neq n} \frac{1}{f(n\nu, t) - f(\ell\nu, t)} \xrightarrow{\nu \to 0^+} P.V. \int_{-\infty}^{\infty} \frac{d\zeta'}{f(\zeta, t) - f(\zeta', t)}. \tag{5.5}$$

Moreover, this approximation shows that representation (5.4) is valid for all time if it is true at $t = t_*$. A rigorous analysis of approximation (5.5) has been performed in the context of vortex sheets in [10]. It is then clear that the pair of equations (5.2a, b) satisfy the following property.

PROPERTY 5.1. *The continuum limit of the CDS and the pole expansion is the parametric system of integro-differential equations defined for any x such that $\forall \zeta \in \mathbb{R}$, $x \neq f(\zeta, t)$, by*

$$u(x,t) = \frac{x}{t} - \int_{-\infty}^{\infty} \frac{d\zeta'}{x - f(\zeta', t)},$$
$$\frac{\partial f}{\partial t}(\zeta, t) = \frac{f(\zeta, t)}{t} - P.V. \int_{-\infty}^{\infty} \frac{d\zeta'}{f(\zeta, t) - f(\zeta', t)}.$$

This property also can be expressed as

$$\frac{\partial f}{\partial t}(\zeta, t) = \frac{f(\zeta, t)}{t} - f(\zeta, t)\, P.V. \int_{-\infty}^{\infty} \frac{d\zeta'}{f^2(\zeta, t) - f^2(\zeta', t)} \tag{5.6}$$

and

$$u(x,t) = \frac{x}{t} - x \int_{-\infty}^{\infty} \frac{d\zeta'}{x^2 - f^2(\zeta', t)}, \qquad x \neq f(\zeta, t). \tag{5.7}$$

Equation (5.7) defines the branch cuts of the inviscid solution as the set of complex x-points for which $x = f(\zeta, t)$, while equation (5.6) defines the dynamics of these branch cuts.

Since the poles are located on the imaginary axis, one can make the additional assumption that $f(\zeta, t) = i\, g(\zeta, t)$, where $g(\zeta, t) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$. Then from (5.7) we find

$$u(x,t) = \frac{x}{t} - 2x \int_0^{\infty} \frac{d\zeta'}{x^2 + g^2(\zeta', t)}, \qquad x \neq i\, g(\zeta, t). \tag{5.8}$$

We then define the density function $\rho(z, t)$ as

$$\rho(z,t) = \frac{1}{g_\zeta(\zeta, t)}, \tag{5.9}$$

where $z = g(\zeta, t)$. Then $d\zeta' = \rho(z', t)\, dz'$ and we introduce this change of variable in (5.8) to obtain

$$u(x,t) = \frac{x}{t} - 2x \int_0^{\infty} \frac{\rho(z;t)}{x^2 + z^2}\, dz.$$

As a converse to the procedure of section 3, case (i) $t = t_*$, the pole positions at t_* can be recovered from the cumulative distribution function σ (see Definition 2.1):

$$\sigma(\beta; \beta_{\max}, t) = \int_0^\beta \rho(\xi; t)\, d\xi \quad \text{for } |\beta| \le \beta_{\max} \tag{5.10}$$

and, in particular, for $t = t_*$, $\rho(\beta; t_*) = \frac{2\sqrt{3}}{\pi}(4t_*)^{-4/3}|\beta|^{1/3}$, so

$$\begin{aligned} \sigma(\beta; \beta_{\max}, t_*) &= \int_0^{\beta_*} \frac{2\sqrt{3}}{\pi}(4t_*)^{-4/3}\xi^{1/3}\, d\xi \\ &= \left(\frac{2\pi}{3\sqrt{3}}\right)^{-1}\left(\frac{\beta_*}{4t_*}\right)^{4/3}. \end{aligned}$$

Inverting this relation in terms of β_*, we find that

$$\beta_* = 4t_*\left(\frac{2\pi}{3\sqrt{3}}\sigma_*\right)^{3/4}. \tag{5.11}$$

In order to recover the correct discretization, it suffices to choose

$$\sigma_* = \sigma_n(t_*, \nu) = 2\nu\mu_n\left(\frac{2\pi}{3\sqrt{3}}\right)^{-1}, \tag{5.12}$$

where $\mu_n = \frac{2\pi}{3\sqrt{3}}(n - \frac{1}{3}) + \mathcal{O}(1/n)$ as $n \to +\infty$ (see [18, section 4.1]). Therefore,

$$\sigma_* = 2\nu(n - 1/3) + \mathcal{O}(1/n) \quad \text{as } n \to +\infty, \tag{5.13}$$

and

$$\beta_* = \beta_n(t_*, \nu) = 4t_*\,(2\nu\mu_n)^{3/4}. \tag{5.14}$$

Similar computations can be found in [21].

6. Uniform asymptotic expansion in $(0, t_*]$ of the spatial Fourier transform of the inviscid solution $\hat{u}(k, t)$ as $k \to +\infty$. The analyticity properties of the inviscid solution also can be analyzed by describing the asymptotic behavior of its Fourier transform (see [13, 20]). We find a uniform asymptotic expansion as $k \to +\infty$ of the Fourier transform of the inviscid solution in a neighborhood of $t = t_*$, where two second-order branch points $\pm x_s(t)$ coalesce into a third-order branch point at the origin $x_s(t_*) = 0$. Thus, we clarify the seemingly discontinuous change of behavior of $\hat{u}(k, t)$ at t_* presented in [13]. This result is resumed in the following theorem.

THEOREM 6.1. *The uniform asymptotic expansion of the Fourier transform of the inviscid solution for $0 < t \le t_*$ is*

$$\hat{u}(k, t) = \mathcal{C}_0 \cdot (tk)^{-4/3} Ai\left[(-3ikx_s(t)/2)^{2/3}\right]\left(1 + \mathcal{O}\left(k^{-1}\right)\right) \quad \text{as } k \to +\infty.$$

Thus from the asymptotic property of the Airy function and its value at the origin $Ai(0)$ we have

$$\hat{u}(k, t) \sim \begin{cases} \mathcal{C}_1(t) \cdot (t_* - t)^{-1/4}\, k^{-3/2}\, e^{-k\,|x_s(t)|} & 0 < t < t_* \\ \mathcal{C}_2 \cdot (t_* k)^{-4/3} & t = t_* \end{cases} \quad \text{as } k \to +\infty.$$

Proof. In [13], Fournier and Frisch derive the asymptotic behavior of the inviscid solution via the so-called Fourier–Lagrangian (F–L) representation, which is valid up to the time where the relation $x(x_0,t) = x_0 + tu_0(x_0)$ is invertible, i.e., up to $t_* = -(\inf_{x_0} u_0'(x_0))^{-1}$. There is a discontinuous change in the behavior of $\hat{u}(k,t)$ in $k^{-3/2}\exp(-k|x_s(t)|)$ before t_* to $(t_*k)^{-4/3}$ at t_*, which arises from the fact that the two saddle points of multiplicity 1 for $0 < t < t_*$ coalesce at the origin to form a saddle point of multiplicity 2 at $t = t_*$. The F–L representation is found by changing variables from the Eulerian coordinate to the Lagrangian coordinate, followed by an integration by parts:

$$\begin{aligned}\hat{u}(k,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}u(x,t)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx(x_0,t)}u_0(x_0)\frac{\partial x}{\partial x_0}dx_0\\ &= \frac{1}{\sqrt{2\pi}ik}\int_{-\infty}^{\infty} e^{-ikx(x_0,t)}u_0'(x_0)\,dx_0 \qquad (k\neq 0).\end{aligned}$$

For $u_0(x) = 4x^3 - x/t_*$, $x(x_0,t) = 4tx_0^3 + x_0(1 - t/t_*)$, we find

$$\hat{u}(k,t) = \frac{1}{\sqrt{2\pi}ik}\int_{-\infty}^{\infty}\exp\left\{-12ikt\left(\frac{x_0^3}{3} - \frac{\alpha}{6}\,x_0\right)\right\}u_0'(x_0)\,dx_0,$$

where $2\alpha = 1/t_* - 1/t$. Let $\lambda = -12ikt$; then we are interested in finding the behavior of $\hat{u}(k,t)$ as $k\to+\infty$, that is, the behavior as $\lambda\to\infty$ of the integral

$$\int_{-\infty}^{\infty}\exp\left\{\lambda\left(\frac{x_0^3}{3} - \frac{\alpha}{6}\,x_0\right)\right\}u_0'(x_0)\,dx_0. \tag{6.1}$$

The saddle points of the integrand occur when $\partial x/\partial x_0 = 0$; thus

$$x_0 = x_0^{\pm}(t) = \pm\sqrt{\frac{\alpha}{6}} = \pm\sqrt[3]{\frac{x_s(t)}{8t}} \Rightarrow x\left(x_0^{\pm}(t)\right) = \pm x_s(t).$$

At $t = t_*$, $x_0^{\pm}(t_*) = 0$, and the two saddle points of multiplicity 1 have coalesced into a saddle point of multiplicity 2 at the origin. Let

$$f(x_0) = \frac{x(x_0,t)}{12t} = \frac{x_0^3}{3} - \frac{\alpha}{6}\,x_0,$$

and recall that $x_s(t) = t(2\alpha/3)^{3/2}$ (see (C.4)). Then

$$f(x_0^{\pm}(t)) = -\frac{2}{3}x_0^{\pm}(t)^3 = \mp\frac{x_s(t)}{12t}.$$

We introduce the coefficients

$$\begin{aligned}\zeta^{3/2} &= \frac{3}{4}\left(f(x_0^-(t)) - f(x_0^+(t))\right) = \frac{3}{2}f(x_0^-(t)) = \frac{x_s(t)}{8t},\\ \eta &= \frac{1}{2}\left(f(x_0^-(t)) + f(x_0^+(t))\right) = 0,\end{aligned}$$

which arise in the construction of a uniform asymptotic expansion of an integral with two coalescing saddle points. The integral defined in (6.1) is already in a format appropriate for such a derivation. Indeed, it is an integral of the form

$$I(\lambda;\zeta,\eta) = \int_{\mathcal{C}}\exp\left\{\lambda\left(u^3/3 - \zeta u + \eta\right)\right\}\phi_0(u)\,du,$$

where $\lambda \to \infty$. Thus the 1–1 analytic transformation $x_0 \leftrightarrow u$ given by the equation $f(x_0) = u^3/3 - \zeta u + \eta$ is simply the identity $x_0 \equiv u$. Therefore, the time-uniform asymptotic expansion of the spatial Fourier transform of the solution $\hat{u}(k,t)$ is immediately found in terms of the Airy function and its derivative (cf. [11, 23, section VII-4]). Let

$$a_0 = \frac{1}{2}\left[u_0'(\zeta^{1/2}) + u_0'(-\zeta^{1/2})\right], \qquad b_0 = \frac{1}{2\zeta^{1/2}}\left[u_0'(\zeta^{1/2}) - u_0'(-\zeta^{1/2})\right],$$

then

$$\hat{u}(k,t) = \frac{e^{-\lambda\eta}}{\sqrt{2\pi}ik}\cdot 2\pi i\left[\frac{\mathrm{Ai}\left[\lambda^{2/3}\zeta\right]}{\lambda^{1/3}}(a_0 + \mathcal{O}(1/\lambda)) + \frac{\mathrm{Ai}'\left[\lambda^{2/3}\zeta\right]}{\lambda^{2/3}}(b_0 + \mathcal{O}(1/\lambda))\right]$$

as $\lambda(k) = -12ikt \to \infty$. Since $\eta = 0$, $\zeta^{3/2} = x_s(t)/8t$, $\zeta^{1/2} = x_0^+(t)$, $u_0'(x_0) = 12x_0^2 - 1/t_*$, $a_0 = u_0'(x_0^\pm(t)) = 1/t$, $b_0 = 0$, and we obtain the asymptotic behavior of $\hat{u}(k,t)$ as $k \to +\infty$ uniform in a compact interval containing $t = t_*$:

$$\hat{u}(k,t) = \mathcal{C}_0 \cdot (tk)^{-4/3}\mathrm{Ai}\left[(-3ikx_s(t)/2)^{2/3}\right]\left(1 + \mathcal{O}\left(k^{-1}\right)\right) \quad \text{as } k \to +\infty, \tag{6.2}$$

where $\mathcal{C}_0$ is an appropriate numerical constant. Note that (6.2) can be obtained without recourse to this method by a classical asymptotic analysis in which one would express the expansion in terms of the Airy function. We choose this derivation due to the simplicity of its construction. For $0 < t < t_*$, using the fact that

$$\mathrm{Ai}(z) = \frac{e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}z^{1/4}}\left(1 + \mathcal{O}\left(z^{-3/2}\right)\right) \quad \text{as } z \to \infty \text{ in } |\arg z| < \pi$$

and for $t = t_*$, and by evaluating $\mathrm{Ai}(0) = 3^{-2/3}/\Gamma(2/3)$, we obtain the asymptotic behavior of $\hat{u}(k,t)$ as $k \to +\infty$ for $0 < t \le t_*$ described in the second part of Theorem 6.1, in which $\mathcal{C}_1(t)$ is a constant depending on t and $\mathcal{C}_2$ is a numerical constant. These expansions are consistent with the fact that the Fourier transform of an analytic function with a branch-point singularity at $x_s = \Re x_s + i\Im y_s$ of the form (cf. [20])

$$v(z) \sim (z - x_s)^\mu, \qquad \mu \notin \mathbb{Z},$$

has an asymptotic behavior of the form

$$\hat{v}(k) \sim k^{-(\mu+1)}e^{-k\Im x_s}e^{ik\Re x_s} \quad \text{as } k \to +\infty.$$

Note that the expansion for $t > t_*$ obtained from (6.2) yields the incorrect behavior $|\hat{u}(k,t)| \sim \mathcal{C}_1(t)\cdot(t - t_*)^{-1/4}k^{-3/2}$, which is valid only for moderate wave numbers of the form $1 \ll k \le 1/|x_s|$. This is due to the fact that the formal F–L representation is no longer valid beyond t_*. The correct behavior after t_* of the form $|\hat{u}_I(k,t)| \sim \mathcal{C}_3(t)\cdot(t - t_*)^{1/2}k^{-1}$ for $k > 1/|x_s|$ which reflects the presence of a shock then must be obtained by following the work of Fournier and Frisch in [13]. The two expansions agree when $k \simeq 1/|x_s|$, giving a behavior of the form $|\hat{u}(k,t)|, |\hat{u}_I(k,t)| \sim \mathcal{C}_4(t)\cdot(t - t_*)^2$ for t close to t_* (see [13]).

Appendix A. On the generic nature of the initial data. Caflisch et al. characterize geometrically generic singularities for nonlinear hyperbolic systems in [9] in the following way: given a PDE and its initial data, a singularity is generic if, under perturbation of the "initial data," the singularity is of one of the stable types, namely, a fold corresponding to a square-root branch point in z for each t or a cusp corresponding to a cube root branch point which occurs when the two square root branch points collide. They show that these are the only stable singularity types for the inviscid BE. Loosely, they define stability as the property that under perturbation of the initial data, the perturbed solution will have the same singularity type as the original problem, i.e., either a fold or a cusp. Note that the formation of a cube root singularity must stem from a "tangential" collision of the square root branch points, i.e., one where the branch points travel at the same characteristic speed. In case of a "nontangential" collision of square root branch points travelling at different characteristic speeds, the resulting singularity remains a square root branch point. For more details see [9].

Fournier and Frisch characterized generic singularities and corresponding generic initial data for the inviscid BE in [13]. This description is based on a local analysis of the singularity and takes into account the Gallilean invariance of the PDE and its invariance under translation of the reference frame. This was reformulated in Bessis and Fournier's first paper [4].

Appendix B. Cardan's formula. The roots of a cubic polynomial are given by the well-known formula of Cardan (cf. [1, section 3.8.2]). We state this formula to clarify the choices that are made in choosing the branches of the algebraic functions which define the saddle points in the expansions: let λ, a, b, $c \in \mathbb{C}$; then the roots of the equation

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \tag{B.1}$$

are obtained by setting

$$A = a/3, \quad B = b/3, \quad \alpha = A^2 - B, \quad \zeta = 2A^3 - 3AB + c.$$

Let $\lambda = x - A$. Then (B.1) becomes

$$x^3 - 3\alpha x + \zeta = 0. \tag{B.2}$$

Let $\omega = e^{2\pi i/3}$ be a cube root of unity; then the three roots of (B.2) are

$$\begin{cases} x_0 = \omega\mathcal{A} + \omega^2\mathcal{B}, \\ x_1 = \omega^2\mathcal{A} + \omega\mathcal{B}, \\ x_2 = \mathcal{A} + \mathcal{B}, \end{cases} \qquad \text{where} \qquad \begin{cases} \Delta = (\zeta/2)^2 - \alpha^3, \\ \mathcal{A} = \sqrt[3]{-\zeta/2 + \sqrt{\Delta}}, \\ \mathcal{B} = \sqrt[3]{-\zeta/2 - \sqrt{\Delta}}. \end{cases} \tag{B.3}$$

After choosing a branch for $\mathcal{A}$, one must choose the corresponding branch for $\mathcal{B}$ so that $\mathcal{A} \cdot \mathcal{B} = \alpha^3$. If α and ζ are real, then there are three possibilities depending on the sign of the real discriminant Δ:

(i) $\Delta < 0:\ \mathcal{A}, \mathcal{B} \in \mathbb{C},\ \mathcal{A} = \overline{\mathcal{B}}, x_0, x_1, x_2 \in \mathbb{R}$;

(ii) $\Delta = 0:\ \mathcal{A} = \mathcal{B} \in \mathbb{R},\ x_0 = x_1 = -\frac{x_2}{2} \in \mathbb{R}$;

(iii) $\Delta > 0:\ \mathcal{A}, \mathcal{B} \in \mathbb{R}, x_0 = \overline{x_1} \in \mathbb{C},\ x_2 \in \mathbb{R}$.

Case (iii), which yields two conjugate roots, is the only instance when we can expect to have two equally relevant saddle points, thus allowing for some cancellation in the asymptotic expansion. The relevant roots x_0 and x_1, after separation of real and imaginary parts, are given by

$$x_0 = \overline{x_1} = -\frac{1}{2}(\mathcal{A} + \mathcal{B}) + i\frac{\sqrt{3}}{2}(\mathcal{A} - \mathcal{B}). \tag{B.4}$$

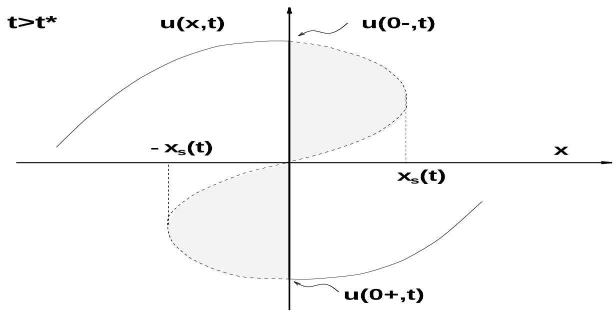

FIG. C.1. *Shock, multivaluedness, branch points, and Maxwell's equal area rule for $t > t_*$.*

Appendix C. Inviscid solution ($\nu = 0$). The inviscid BE states that the velocity of a fluid particle is conserved along certain trajectories, namely, the characteristic lines

$$\dot{x} = \frac{dx}{dt}(t) = u(x(t), t) \tag{C.1}$$

in the (x, t) plane. The implicit solution obtained by the method of characteristics reflects the conservation of the velocity along these special curves:

$$\begin{cases} u = u(x,t) = u_0(x_0(x,t)), \\ x = x_0 + t\, u_0(x_0(x,t)). \end{cases} \tag{C.2}$$

A fluid particle originally at a (Lagrangian) position x_0 in space will be at a new (Eulerian) position x after a certain time t with the same velocity along this line. Let $U = x_0$. Then, substituting $u_0(x) = 4x^3 - x/t_*$ in (C.2), we find that U satisfies the cubic equation

$$U^3 - \frac{\alpha}{2}U - \frac{x}{4t} = 0, \qquad \alpha = \frac{t - t_*}{2tt_*}. \tag{C.3}$$

This defines a three-sheeted Riemann surface for the solution with a third-order branch point at infinity and two opposite second-order branch points at $\pm x_s(t)$ defined by

$$x_s(t) = t\,(2\alpha/3)^{3/2} = i\,(3t_*)^{-3/2}(t_* - t)^{3/2} t^{-1/2}. \tag{C.4}$$

The envelope of the characteristic lines is the branch point since $0 = \frac{\partial x}{\partial x_0} \Rightarrow x\left(x_0^{\pm}(t)\right) = \pm x_s(t)$. The solution is therefore

$$U(x,t) = \begin{cases} (8t)^{-1/3}\left\{\sqrt[3]{x + \sqrt{x^2 - x_s^2}} + \sqrt[3]{x - \sqrt{x^2 - x_s^2}}\right\} & t \neq t_*, \\ \sqrt[3]{\frac{x}{4t_*}} & t = t_*. \end{cases} \tag{C.5}$$

Note the particular (real) values of $u(x, t)$ at the shock at $x = 0$ already found in (3.13):

$$u(0^{\pm}, t) = -U(0^{\pm}, t)/t = \begin{cases} \mp\frac{1}{2}\,(t - t_*)^{1/2} t^{-3/2} t_*^{-1/2} & t \geq t_*, \\ 0 & t < t_*. \end{cases} \tag{C.6}$$

The topology of the three-sheeted Riemann surface given by (C.5) and the interpretation of the shock as the permutation of two Riemann sheets has been fully explained by Bessis and Fournier in [3].

Appendix D. Generalization of the initial data to $u_0(x) = 2nx^{2n-1} - x/t_*$, $n \geq 2$. Although the inviscid singularity resulting from a polynomial of arbitrary odd order of the form $u_0(x) = 2nx^{2n-1} - x/t_*$, $n \in \mathbb{N}$, $n \geq 2$, is no longer generic (see [13, p. 707]), it is still interesting to describe the behavior of the inviscid solution and the related asymptotic density of poles. We first describe the inviscid solution and its branch points: substituting $u_0(x)$ in (C.2), we find that U is a root of the polynomial of degree $2n-1$. Let

$$\alpha_n(t) = \frac{t - t_*}{ntt_*} \quad \text{and} \quad P_n(U) = U^{2n-1} - \frac{\alpha_n(t)}{2}\, U - \frac{x}{2nt} = 0.$$

Let $U_s(t)$ satisfy

$$0 = \frac{\partial x}{\partial U}(U_s(t)) = P_n'(U_s(t)) \Longrightarrow U_s(t) = \left(\frac{\alpha_n(t)/2}{2n-1}\right)^{1/(2n-2)}.$$

The $2n-2$ branch points of the inviscid solution are then given by

$$x_s(t) = x(U_s(t)) = 2ntU_s\left(U_s^{2n-2} - \alpha_n(t)/2\right) = \mathcal{C}_n \cdot (t - t_*)^{\frac{2n-1}{2n-2}} \cdot t_*^{-\frac{2n-1}{2n-2}} \cdot t^{-\frac{1}{2n-2}},$$

where $\mathcal{C}_n = -(2n-2)(2n-1)^{\frac{2n-1}{2n-2}}(2n)^{-\frac{1}{2n-2}}$. Notice that the $2n-2$ branch points coalesce at the origin at t_*. Since $U(0^\pm, t) = \pm\pi t\rho(0;t) = \pm(\alpha_n(t)/2)^{\frac{1}{2n-2}}$ (cf. (3.9)), the (real) value of the inviscid solution at the origin (shock) is given by

$$u(0^\pm, t) = -U(0^\pm, t)/t = \mp\pi\rho(0;t) = \begin{cases} \mp(t - t_*)^{\frac{1}{2n-2}}(2nt_*)^{\frac{-1}{2n-2}}t^{-\frac{2n-1}{2n-2}} & t \geq t_*, \\ 0 & t < t_*. \end{cases}$$

The limiting pole density and inviscid limit are obtained using results from [17], where it is shown that the relevant saddle points in the asymptotic analysis of $E_\nu(i\beta, t_*)$ are

$$z_0(\beta; t_*) = \exp\left(\frac{i\pi}{4n-2}\right)\left(\frac{\beta}{2nt_*}\right)^{\frac{1}{2n-1}}, \quad z_1 = -\overline{z_0}.$$

Since

$$\rho(\beta; t_*) = \frac{\Re z_0(\beta; t_*)}{\pi t_*}$$

and further results in [17] concern the asymptotic behavior of the zeros $\mu_{k,n}$ of $\mathcal{F}_n(\mu) = \int_{-\infty}^{\infty} e^{\mu(2niz - z^{2n})}dz$, we have the following result.

PROPERTY D.1. *For any integer $n \geq 2$, the density of poles at the shock time t_* arising from the initial data $u_0(x) = 2nx^{2n-1} - x/t_*$ is*

$$\rho(\beta; t_*) = \frac{1}{\pi}\cos\left(\frac{\pi}{4n-2}\right)\left(\frac{\beta}{2nt_*^{2n}}\right)^{\frac{1}{2n-1}}.$$

The density at the origin ($z = 0$) is

$$\rho(0;t) = \begin{cases} \frac{1}{\pi t}\left(\frac{t-t_*}{2ntt_*}\right)^{\frac{1}{2n-2}} & t > t_*, \\ 0 & t \le t_*. \end{cases}$$

Moreover at t_*, the kth ordered pole of the solution for $\nu > 0$ is located at

$$a_{k,n}(t_*, \nu) = i \cdot 2nt_* \left(2\nu\mu_{k,n}\right)^{\frac{2n-1}{2n}},$$

where the positive coefficients $\mu_{k,n}$ are asymptotically given by

$$\mu_{k,n} = \frac{\pi}{4n-2} \sec\left(\frac{\pi}{4n-2}\right)\left(\frac{n-1}{2n-1} + 1 + 2k\right) + \mathcal{O}\left(\frac{1}{k}\right) \quad \textit{as } k \to +\infty.$$

Higher order approximations of $\mu_{k,n}$ are provided in [17]. At $t = t_*$, the inviscid solution can be found via the complex-valued limiting pole density $\rho(\beta; t_*)$ and the pole expansion as in (3.11) or via the characteristic equation as in Appendix C. The resulting singularity is a branch point of order $2n-1$, which arises from the coalescence of $n-1$ pair(s) of conjugate branch points of order $2n-2$ (see [13, p. 707]):

$$u(x, t_*) = \frac{x}{t_*} - \left(\frac{x}{2nt_*^{2n}}\right)^{\frac{1}{2n-1}}.$$

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